5. **Geometric Modeling**

- Types of Curves and Their Mathematical Representation
- Types of Surfaces and Their Mathematical Representation
- Types of Solids and Their Mathematical Representation
- CAD/CAM Data Exchange
TYPES OF CURVES AND THEIR MATHEMATICAL REPRESENTATIONS

- **Wireframe Model** (2D in 1960s for drafting, 3D in 1970s)
- **Wireframe Entities**
  - analytic entities (points, lines, arcs and circles, fillets and chamfers, and conics)
  - synthetic entities (splines and Bezier curves)
  - methods of defining points: \( P(x,y,z) \), \( P(r,\theta,z) \), \( P(x+\Delta x,y+\Delta y,z+\Delta z) \), \( \ldots \), end points of existing entity, center point, intersection of two entities.
  - methods of defining lines: between two points, parallel to axis, parallel or perpendicular to a line, tangent to entity
  - methods of defining arcs and circles: center and radius, three points, center and a point, a radius and tangent to a line passing through a point.
  - methods of defining ellipses and parabolas: ellipse (center and axes lengths, four points, two conjugate diameters), parabola (vertex and focus, three points).
  - methods of defining synthetic curves: cubic spline (a set of data points and end slopes), Bezier curves (a set of data points), B-spline curves (interpolate a set of data points with local control possible).
Curve Representation: Two types of representation are parametric and non-parametric representation. In parametric representation all variables (i.e., coordinates) are expressed in terms of common parameters. For example, a point can be expressed with respect to a parameter as

\[ P(u) = [x(u), y(u), z(u)], \quad u_{\text{min}} \leq u \leq u_{\text{max}} \]

Non-parametric representation is the conventional representation as

\[ P = [x, y(x), z(x)] \]

Ex. Non-parametric form of a circle: \( x^2 + y^2 = r^2 \), parametric form: \( P(u) = [r \cos 2\pi u, r \sin 2\pi u], \ 0 \leq u \leq 1 \). This form can be used to find slopes at a certain angle for example.
The following list shows most of the analytic curve that are used in CAD/CAM system for part design and modeling.

- Lines
- Circles
- Ellipses
- Parabolas
- Hyperbolas
- Conic Curves
A line between two points $P_1$ and $P_2$ can be expressed with respect to a parameter.

$$ P = P_1 + u(P_2 - P_1) $$

A circle for a center and the radius can be written as

$$ x = x_c + R \cos u $$
$$ y = y_c + R \sin u, \quad 0 \leq u \leq 2\pi $$
$$ z = z_c $$
An **ellipse** with a center and major and minor axes of 2A and 2B can be expressed as:

\[
x = x_c + A \cos u \\
y = y_c + B \sin u, \quad 0 \leq u \leq 2\pi \\
z = z_c
\]
A parabola in the local coordinate system that is parallel to the global coordinate system with the vertex \((x_v, y_v)\) and the focal distance \(A\) from the vertex in a plane \(z = z_v\) is given by

\[
\begin{align*}
x &= x_v + x' = x_v + A u^2 \\
y &= y_v + y' = y_v + 2 A u, \quad -\infty \leq u \leq \infty \\
z &= z' = z_v
\end{align*}
\]
When the parabola is inclined at angle $\theta$ relative to global $x$-axis, the equation for the parabola is given by

\begin{align*}
    x &= x_v + x'_\cos \theta - y'_\sin \theta = x_v + A u^2 \cos \theta - 2 A u \sin \theta \\
    y &= y_v + x'_\sin \theta + y'_\cos \theta = y_v + A u^2 \sin \theta + 2 A u \cos \theta, \quad -\infty \leq u \leq \infty \\
    z &= z_v
\end{align*}
Example 4.2.1.1. Determine the equation above for given three points, \( P_1(5,10) \), \( P_2(3,4) \), and \( P_3(12,1) \). The parabola is inclined at 30° relative to global x-axis. Plot the curve by varying the parameter \( u \) from -5 to 5.

**Solution:** Applying the six conditions (two for each point), the equations to solve are obtained as

\[
\begin{align*}
5 &= x_v + Au_1^2 \cos \frac{30}{180} \pi - 2Au_1 \sin \frac{30}{180} \pi \\
10 &= y_v + Au_1^2 \sin \frac{30}{180} \pi + 2Au_1 \cos \frac{30}{180} \pi \\
3 &= x_v + Au_2^2 \cos \frac{30}{180} \pi - 2Au_2 \sin \frac{30}{180} \pi \\
4 &= y_v + Au_2^2 \sin \frac{30}{180} \pi + 2Au_2 \cos \frac{30}{180} \pi \\
12 &= x_v + Au_3^2 \cos \frac{30}{180} \pi - 2Au_3 \sin \frac{30}{180} \pi \\
1 &= y_v + Au_3^2 \sin \frac{30}{180} \pi + 2Au_3 \cos \frac{30}{180} \pi
\end{align*}
\]

This is a set of nonlinear equations. The solution and the graph are

\[
\begin{align*}
x_v &= \frac{-1218761}{403520} \sqrt{3} + \frac{3465571}{403520} \\
y_v &= \frac{849279}{403520} + \frac{610667}{1210560} \sqrt{3} \\
A &= \frac{129}{160} + \frac{11}{32} \sqrt{3}
\end{align*}
\]
Example 4.2.1.2. Determine the parabola for given three points, P₀(5,10), P₁(3,4), and P₂(12,1). P₀ is the vertex of the parabola. Plot the curve by varying the parameter \( u \) from -5 to 5.

Solution: Applying the six conditions (two for each point), the equations to solve become

\[
\begin{align*}
A u_1^2 \cos \theta - 2 A u_1 \sin \theta &= x_1 - x_v \\
A u_1^2 \sin \theta + 2 A u_1 \cos \theta &= y_1 - y_v \\
A u_2^2 \cos \theta - 2 A u_2 \sin \theta &= x_2 - x_v \\
A u_2^2 \sin \theta + 2 A u_2 \cos \theta &= y_2 - y_v
\end{align*}
\]

(4.2.1.7)

Solving these equations for \( A, \theta, u_1, \) and \( u_2 \) yields the solutions as

\[
A = 0.543, \ \theta = -4.372, \ \ u_1 = -4.372, \ \text{and} \ \ u_2 = 3.835
\]
A hyperbola with the center \((x_v, y_v)\) and the distances \(A\) and \(B\) in a plane \(z = z_v\) in the figure below can be expressed as

\[
\begin{align*}
    x &= x_v + A \cosh u \\
    y &= y_v + B \sinh u, \quad -\infty \leq u \leq \infty \\
    z &= z_v
\end{align*}
\]

(4.2.1.8)

Figure 4.2.1.1. A hyperbola and its asymptotes
Example 1.3. Determine the hyperbola for given three points, P1(2,1), P2(8,9), P3(12,1), and A=1. B is the vertex of the hyperbola. Plot the curve by varying the parameter u from -2 to 2.

Solution: Applying the seven conditions (two for each point), the equations to solve become...

The solution is B = 0.503, θ = 0.464, u1 = 2.881, and u2 = -2.881. The graph of the curve is shown as...

(4.2.1.9)
The most general form of planar quadratic curves is conic curves or conic sections that include the previously covered curves; lines, circles, ellipses, parabolas, and hyperbolas. The general implicit nonparametric quadratic equation that describes the planar conic curve has five coefficients and naturally needs five conditions to complete it.

The conic parametric equation can be described if five conditions are specified appropriately. One case is specifying five points on the curve.

\[
L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0
\]

\[
L_1L_2 = 0, \quad L_3L_4 = 0
\]

\[
L_1L_2 + aL_3L_4 = 0
\]
**Example 4.2.1.4.** Find the equation of a conic curve defined by five points \( P_1(0,0) \), \( P_2(1,5) \), \( P_3(2,3) \), \( P_4(3,-1) \) and \( P_5(4,-3) \). The \( z \)-planes coincide between WCS and MCS. Also, plot the curve.

**Solution:** The equations of straight lines are given by

\[
L_1(x, y) = (y_1 - y_0)x - (x_1 - x_0)y + x_1y_0 - y_1x_0 = 0 \\
L_2(x, y) = (y_4 - y_3)x - (x_4 - x_3)y + x_4y_3 - y_4x_3 = 0 \\
L_3(x, y) = (y_3 - y_0)x - (x_3 - x_0)y + x_3y_0 - y_3x_0 = 0 \\
L_4(x, y) = (y_4 - y_1)x - (x_4 - x_1)y + x_4y_1 - y_4x_1 = 0
\]

From these equations two quadratic curves are obtained as

\[
L_1L_2 = 0, \quad L_2L_4 = 0
\]

Combining these two equations with a constant as

\[
L_1L_2 + \alpha L_3L_4 = 0
\]

and applying the condition that the curve must pass through \( P_2 \) leads to the value of the constant.

\[
\alpha = \frac{L_1(x_2, y_2)L_2(x_2, y_2)}{L_3(x_2, y_2)L_4(x_2, y_2)} = 0.636
\]
Parametric Representation of Synthetic Curves

- Hermite Cubic Splines
- Bezier Curves: cubic curve with four control points.
- B-Splines: general case of Bezier’s curve (non-uniform)
- Rational Curves (algebraic ratio of two polynomials)
- NURB (non-uniform rational B-spline) curve combines all features of previous curves
Hermite Cubic Splines

The parametric representation of a cubic spline segment between two points $P_0$ at $u = 0$ and $P_1$ at $u = 1$ can be written as

$$P(u) = \sum_{j=0}^{3} C_j u^j = C_0 + C_1 u + C_2 u^2 + C_3 u^3 = U^T C, \quad 0 \leq u \leq 1$$

where $U = [u^3, u^2, u, 1]^T$ and $C = [C_3, C_2, C_1, C_0]^T$.

The coefficient vector $C$ can be determined by applying the end conditions ($P_0$, $P_0'$ at $u = 0$ and $P_1$, $P_1'$ at $u = 1$).

$$P_0 = C_0$$
$$P_0' = C_1$$
$$P_1 = C_0 + C_1 + C_2 + C_3$$
$$P_1' = C_1 + 2C_2 + 3C_3$$

Solving these equations for the coefficients yields

$$C_0 = P_0$$
$$C_1 = P_0'$$
$$C_2 = 3(P_1 - P_0) - 2(P_0' - P_1')$$
$$C_3 = 2(P_0 - P_1) + (P_0' + P_1')$$
Thus, the equation becomes

\[ P(u) = (2u^3 - 3u^2 + 1)P_0 + (-2u^3 + 3u^2)P_1 + (u^3 - 2u^2 + u)P_0' + (u^3 - u^2)P_1' \]

The tangent vector can be obtained by taking a derivative with respect to \( u \).

Further, this equation can be written in a matrix form as

\[ P(u) = U^T M_H V \]

where \( U^T = [u^3, u^2, u, 1] \), \( M_H \) is the Hermite matrix and \( V \) is the geometry vector.

\[ M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad V = [P_0, P_1, P_0', P_1']^T \]
Example 5.2.1.7. Plot the spline curve defined by two points and their slopes, $P_1(0,0)$, $P_2(4,2)$, $P_1'(5,6)$, and $P_2'(-3,5)$.

Solution: The curve from (5.2.2.4) is given in the first curve. The second curve is for same points except $P_1'(-5,6)$. The second figure is drawn with three points with $P_3(7,1)$ and $P_3'(0,0)$. The curve is not really smooth. Usually, certain smooth conditions (i.e., slopes and curvatures are made same from both adjoining curves) are used at joints of curves to make the curve smooth for a cubic spline with several points. Right now, the curvatures of both curves are not same at the joint.
By taking derivatives of the (5.2.2.4), the second derivatives are obtained

\[(5.2.2.12) \quad P''(u) = (12u - 6)P_0 + (-12u + 6)P_1 + (6u - 4)P_0' + (6u - 2)P_1'\]

Using this equation for curvature, the smoothness condition on the curvature from two adjoining curves can be set equal $P_{c1}''(1) = P_{c2}''(0)$. This is two equations, one for $x$ and the other for $y$. Solving this equations each yields the solution $P_2' = \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 4 \\ -3/4 \end{pmatrix}$. Using these values the curve now shows smooth connection.
Example 5.2.1.8. Plot the natural cubic spline curve defined by five points, $P_1(1,4)$, $P_2(4,6)$, $P_3(6,8)$, $P_4(9,7)$, and $P_5(10,7)$.

Solution: In this case the equation (5.2.2.7) at all internal points yield a system of equations.

$$
\begin{align*}
P''_{e_1}(0) &= (0-6)P_1 + (0+6)P_2 + (0-4)P_1' + (0-2)P_2' = 0 \\
P''_{e_2}(1) &= (12-6)P_1 + (-12+6)P_2 + (6-4)P_1' + (6-2)P_2' = (0-6)P_2 + (0+6)P_3 + (0-4)P_2' + (0-2)P_3' = P''_{e_2}(0) \\
P''_{e_3}(1) &= (12-6)P_2 + (-12+6)P_3 + (6-4)P_2' + (6-2)P_3' = (0-6)P_3 + (0+6)P_4 + (0-4)P_3' + (0-2)P_4' = P''_{e_3}(0) \\
P''_{e_4}(1) &= (12-6)P_3 + (-12+6)P_4 + (6-4)P_3' + (6-2)P_4' = (0-6)P_4 + (0+6)P_5 + (0-4)P_4' + (0-2)P_5' = P''_{e_4}(0)
\end{align*}
$$

The matrix form of this equation is

$$
\begin{bmatrix}
4 & 2 \\
2 & 8 & 2 \\
2 & 8 & 2 \\
2 & 8 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
P_1' \\
P_2' \\
P_3' \\
P_4' \\
P_5'
\end{bmatrix}
= -6
\begin{bmatrix}
P_1 - P_2 \\
P_2 - P_3 \\
P_3 - P_4 \\
P_4 - P_5
\end{bmatrix}
\quad (5.2.2.13)
$$
For \( x \) and \( y \) each, this five equations are solved. The solutions are \( x^* = \{27/8, 9/4, 21/8, 9/4, 3/8\} \) and \( y^* = \{99/56, 69/28, 3/8, -27/28, 27/56\} \). The spline curve appears in the following figure on left. The curve on right was obtained with \( P_5(1,7) \).
Another alternative to create curves is to use approximation techniques which produce curves that do not pass through the given data points that are rather used to control the shape of the curves. Approximation techniques are more often preferred over interpolation techniques in curve design due to the added flexibility and the additional intuitive feel.

Bezier curves and surfaces are credited to P. Bezier of the French car firm Regie Renault who developed (about 1962) and used them in his software system called UNISURF which has been used by designers to define the outer panels of several Renault cars.
Figure 5.2.2.1. Cubic Bezier curve with four data points

The curve can be expressed parametrically as

\[ P(u) = \sum_{m=0}^{n} P_m B_{m,n}(u), \quad 0 < u \leq 1 \]

where \( P_m \) is a data point and \( B_{m,n} \) are Bernstein polynomials which are given by

\[ B_{m,n}(u) = \binom{n}{m} u^m (1-u)^{n-m} \]

\[ \binom{n}{m} = \frac{n!}{m!(n-m)!} \]
Example 5.2.2.1. For the given four data points below, find the equation for Bezier curve and also find points on the curve at $u = 0, 1/3, 1/2, 1$.

$$P_0 = [1, 1, 0]^T, \quad P_1 = [2, 4, 0]^T, \quad P_2 = [5, 5, 0]^T, \quad P_3 = [4, 2, 0]^T$$

Solution. Using the equation (5.2.2.9) the curve can be plotted as
Example 5.2.2.2. For the given four data points below, find the equation for Bezier curve and also find points on the curve at $u = 0$, $1/3$, $1/2$, $1$.

$$P_0 = [1, 1]^T, \quad P_1 = [2, 4]^T, \quad P_2 = [5, 5]^T, \quad P_3 = [4, 2]^T$$

Solution. From (5.2.2.10) the points are found as:

$$x(0) = 1, \quad y(0) = 1, \quad x(1/3) = 2.444, \quad y(1/3) = 3.259, \quad x(2/3) = 3.889, \quad y(2/3) = 3.741, \quad x(1) = 6, \quad y(1) = 1$$

The curve is shown in the first figure. The second figure shows another curve that has same starting and ending points.
In contrast to Bezier curves, the theory of B-spline curves **separates** the degree of the resulting curve from the number of the given control points.

The B-spline curve \( P(u) \) for the degree \( k \) defined by \( n + 1 \) control points

\[
P(u) = \sum_{j=0}^{n} B_j^k(u)P_j, \quad 0 \leq u \leq u_{\max}
\]

*partition of unity*

\[
\sum_{j=0}^{n} B_j^k(u) = 1
\]
the recursive property

\[ B_j^k(u) = \frac{u-u_j}{u_j+k-u_j} B_j^{k-1}(u) + \frac{u_{j+k+1}-u}{u_{j+k+1}-u_{j+1}} B_{j+1}^{k-1}(u), \quad k \geq 1 \]

\[ B_j^0(u) = \begin{cases} 
1, & u_j \leq u \leq u_{j+1}, \quad \text{for } j = n \\
0, & \text{otherwise} \\
1, & u_j \leq u < u_{j+1} \\
0, & \text{otherwise} 
\end{cases} \]

open parametric knots

\[ u_j = \begin{cases} 
0, & j < k+1 \\
-n-k+1, & j > n \\
-j-k, & \text{otherwise} 
\end{cases} \]

for \(0 \leq j \leq n + k + 1\)
Knot Values

- number of knots = number of points + degree + 1 \((n_k = n+1+k+1)\) must be satisfied
- The individual knot values are not meaningful by themselves; only the ratios of the difference between the knot values matter.
- To be able to use the parametric knots produced by the coordinates, the values must be in increasing order. Thus, if the curve intersects itself, this cannot be used.
- The number of duplicate values is limited to no more than the degree. Duplicate knot values make the b-spline curve less smooth. At the extreme, a full multiplicity knot in the middle of the knot list means there is a place on the b-spline curve that can be bent into a sharp kink.
Summary of B-spline Basis Function $B_{j}^{k}$

- a polynomial function of degree $k$
- nonnegative for all $j$ and $k$ (nonnegativity)
- non-zero only on $[u_{j}, u_{j+k+1}]$ (local support)
- **at most** $k+1$ basis functions of degree $k$ are non-zero on knot span $[u_{j}, u_{j+1}]$, namely, $B_{j-k}^{k}$, $B_{j-k+1}^{k}$, ..., $B_{j}^{k}$
- sum of all degree $k$ basis functions on knot span $[u_{j}, u_{j+1}]$ is one (partition of unity)
- at a knot of **multiplicity** $m$, basis function $B_{j}^{k}$ is $C^{k-m}$ continuous.
- a composite curve of degree $k$ polynomials with joining knots in $[u_{j}, u_{j+k+1}]$
- $n+1$ control points, $P_{0}, P_{1}, ..., P_{n}$
- $n-2$ basis functions (i.e., cubic polynomial curve segments, $Q_{3}, Q_{4}, ..., Q_{n}$)
- $n-1$ knot points, $t_{3}, t_{4}, ..., t_{n+1}$ (any additional ones needed for computation are set to same values as the nearest)
- $B_{j}^{k}$ is
  - defined over a knot interval $[u_{j}, u_{j+1}]$
  - defined by four of the control points, $P_{j-3}$, ..., $P_{j}$
Notable **benefits** of b-spline include:

- **Independent degree**: the degree is set by user and it has nothing to do with number of data points
- **a single piecewise curve** of a particular degree: there is no need to stitch together separate curves as in interpolation splines such as natural cubic spline
Example 5.2.2.2. For the given four data points below that are same as the previous example, find the equation for cubic B-spline curve.

\[ P_0 = [1, 1, 0]^T, \quad P_1 = [2, 4, 0]^T, \quad P_2 = [5, 5, 0]^T, \quad P_3 = [4, 2, 0]^T \]

Solution. The data are \( n = 3 \) and \( k = 3 \) and the curve is

\[ P(u) = \sum_{j=0}^{n} P_j B_{j-k}^k (u), \quad 0 \leq u \leq n - k + 1 \]

The parametric knots are obtained as \( u_i = \{0, 0, 0, 0, 1, 1, 1\} \) from (5.2.2.14). The range of \( u \) is between 0 and 1.

B-spline functions are given below.

\[ B_0^0 = B_1^0 = B_2^0 = 0, \quad B_3^0 = \begin{cases} 1, & 0 \leq u \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad B_4^0 = B_5^0 = 0 \]
\[ B_0^1(u) = \frac{u-u_0}{u_1-u_0} B_0^0(u) + \frac{u_2-u}{u_2-u_1} B_1^0(u) = \frac{u}{0} B_0^0(u) + \frac{-u}{0} B_1^0(u) = 0 \]

\[ B_1^1(u) = \frac{u-u_1}{u_2-u_1} B_1^0(u) + \frac{u_3-u}{u_3-u_2} B_2^0(u) = \frac{u}{0} B_1^0(u) + \frac{-u}{0} B_2^0(u) = 0 \]

\[ B_2^1(u) = \frac{u-u_2}{u_3-u_2} B_2^0(u) + \frac{u_4-u}{u_4-u_3} B_3^0(u) = \frac{u}{0} B_2^0(u) + \frac{1-u}{1} B_3^0(u) = \frac{1-u}{1} B_3^0(u) \]

\[ B_3^1(u) = \frac{u-u_3}{u_4-u_3} B_3^0(u) + \frac{u_5-u}{u_5-u_4} B_4^0(u) = \frac{u}{1} B_3^0(u) + \frac{1-u}{0} B_3^0(u) = \frac{u}{1} B_3^0(u) \]

\[ B_4^1(u) = \frac{u-u_4}{u_5-u_4} B_4^0(u) + \frac{u_6-u}{u_6-u_5} B_5^0(u) = \frac{u-1}{0} B_4^0(u) + \frac{1-u}{0} B_5^0(u) = 0 \]

\[ B_5^1(u) = \frac{u-u_5}{u_6-u_5} B_5^0(u) + \frac{u_7-u}{u_7-u_6} B_6^0(u) = \frac{u-1}{0} B_5^0(u) + \frac{1-u}{0} B_6^0(u) = 0 \]
\[B_0^2(u) = \frac{u-u_0}{u_2-u_0} B_0^1(u) + \frac{u_3-u}{u_3-u_1} B_1^1(u) = \frac{u}{0} B_0^1(u) + \frac{-u}{0} B_1^1(u) = 0\]

\[B_1^2(u) = \frac{u-u_1}{u_3-u_1} B_1^1(u) + \frac{u_4-u}{u_4-u_2} B_2^1(u) = \frac{u}{0} B_1^1(u) + \frac{1-u}{1} B_2^1(u) = \frac{1-u}{1} B_2^1(u)\]

\[B_2^2(u) = \frac{u-u_2}{u_4-u_2} B_2^1(u) + \frac{u_5-u}{u_5-u_3} B_3^1(u) = \frac{u}{1} B_2^1(u) + \frac{1-u}{1} B_3^1(u)\]

\[B_3^2(u) = \frac{u-u_3}{u_5-u_3} B_3^1(u) + \frac{u_6-u}{u_6-u_4} B_4^1(u) = \frac{u}{1} B_3^1(u) + \frac{1-u}{0} B_4^1(u) = \frac{u}{1} B_3^1(u)\]

\[B_4^2(u) = \frac{u-u_4}{u_6-u_4} B_4^1(u) + \frac{u_7-u}{u_7-u_5} B_5^1(u) = \frac{u-1}{0} B_4^1(u) + \frac{1-u}{0} B_2^1(u) = 0\]

\[B_0^3(u) = \frac{u-u_0}{u_3-u_0} B_0^2(u) + \frac{u_4-u}{u_4-u_1} B_1^2(u) = \frac{u}{0} B_0^2(u) + \frac{1-u}{1} B_1^2(u) = \frac{1-u}{1} B_1^2(u)\]

\[B_1^3(u) = \frac{u-u_1}{u_4-u_1} B_1^2(u) + \frac{u_5-u}{u_5-u_2} B_2^2(u) = \frac{u}{1} B_1^2(u) + \frac{1-u}{1} B_2^2(u)\]

\[B_2^3(u) = \frac{u-u_2}{u_5-u_2} B_2^2(u) + \frac{u_6-u}{u_6-u_3} B_3^2(u) = \frac{u}{1} B_2^2(u) + \frac{1-u}{1} B_3^2(u)\]

\[B_3^3(u) = \frac{u-u_3}{u_6-u_3} B_3^2(u) + \frac{u_7-u}{u_7-u_4} B_4^2(u) = \frac{u}{1} B_3^2(u) + \frac{1-u}{0} B_4^2(u) = \frac{u}{1} B_3^2(u)\]
With the B-spline functions determined above, the curve is completely determined. The curve is same as that in the previous example and shown below.
Parametric Knots

\[ UP(n,k) := \begin{cases} \text{for } i \in 0..n+k+1 & 0 \leq i \leq n+k+1 \\ \text{up}_1 \leftarrow 0 \text{ if } i < k+1 \\ \text{otherwise} \\ \quad \text{up}_1 \leftarrow n-k+1 \text{ if } i > n \\ \quad \text{up}_1 \leftarrow i-k \text{ otherwise} \\ \text{up} \\ \text{upmax} = n-k+1 \end{cases} \]
B-spline functions

\[ B(j, k, n, u, u_p) := \begin{cases} 
  \text{if } k = 0 & \\
  \text{if } j + 1 = n + k + 1 & \\
  \text{if } u \leq u_{p_{j+1}} & \\
  \text{sum} \leftarrow 1 \text{ if } u \geq u_{p_j} & \\
  \text{sum} \leftarrow 0 \text{ otherwise} & \\
  \text{sum} \leftarrow 0 \text{ otherwise} & \\
  \text{otherwise} & \\
  \text{if } u < u_{p_{j+1}} & \\
  \text{sum} \leftarrow 1 \text{ if } u \geq u_{p_j} & \\
  \text{sum} \leftarrow 0 \text{ otherwise} & \\
  \text{sum} \leftarrow 0 \text{ otherwise} & \\
  \text{otherwise} & \\
  \text{sum} \leftarrow \frac{(u - u_{p_j})}{u_{p_{j+k}} - u_{p_j}} \cdot B(j, k - 1, n, u, u_p) \text{ if } u_{p_{j+k}} > u_{p_j} & \\
  \text{sum} \leftarrow \text{sum} + \frac{u_{p_{j+k+1}} - u}{u_{p_{j+k+1}} - u_{p_{j+1}}} \cdot B(j + 1, k - 1, n, u, u_p) \text{ if } u_{p_{j+k+1}} > u_{p_{j+1}}
\end{cases} \]
Input Data

\[ \begin{align*}
\mathbf{x}_p &= \begin{pmatrix} 1 \\ 2 \\ 5 \\ 4 \end{pmatrix} & \mathbf{y}_p &= \begin{pmatrix} 1 \\ 4 \\ 5 \\ 2 \end{pmatrix} \\
\end{align*} \]

\[ n := \text{rows}(\mathbf{x}_p) - 1 \quad k := 3 \quad n = 3 \]

\[ \mathbf{u}_p := \text{UP}(n,k) \quad \mathbf{u}_p^T = (0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1) \]
Range of $u$

$$u := 0, 0.05 .. n - k + 1$$

$$x(u) := \sum_{i=0}^{n} x_p \cdot B(j, k, n, u, u_p)$$

$$y(u) := \sum_{i=0}^{n} y_p \cdot B(j, k, n, u, u_p)$$
**Uniform Quadratic B-splines**

A quadratic B-spline with a uniform knot-vector is a commonly used form of B-spline. The blending function can easily be precalculated, and is equal for each segment in this case.

\[
B_j^2(u) = \begin{cases} 
(1-u)^2 & 
\text{if } 0 \leq u \leq 1 \\
2u(1-u) & 
\text{if } 0 \leq u \leq 1 \\
-u^2 & 
\text{if } 0 \leq u \leq 1 
\end{cases}
\]

In a matrix form, it is

\[
P(u) = \begin{bmatrix} u^2, u, 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{bmatrix}, \quad 0 \leq u \leq 1
\]
Uniform Cubic B-splines

Cubic B-spline with uniform knot-vector is the most commonly used form of B-spline. The blending function can be easily precalculated, and is equal for each segment in this case. In a matrix form, it becomes

\[
B^3_j(u) = \begin{cases} 
(1-u)^3, & 0 \leq u \leq 1 \\
3u(1-u)^2, & 0 \leq u \leq 1 \\
3u^2(1-u), & 0 \leq u \leq 1 \\
u^3, & 0 \leq u \leq 1 
\end{cases}
\]

\[
P(u) = \begin{bmatrix}
-u^3 & u^2 & u & 1 \\
3 & -6 & 3 & 0 \\
-3 & 3 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
P_{i-1} \\
P_i \\
P_{i+1} \\
P_{i+2}
\end{bmatrix}, \quad 0 \leq u \leq 1
4. Rational Curves

These curves are algebraic ratio of two polynomials. The rational form provides a unified representation for conventional free-form curves and surfaces along with conic sections and quadratic surfaces, is invariant under projective transformation, and possesses weights, which can be used to control shape in a manner similar to shape parameters. Rational Bezier curves, rational B-spline curves, rational conic curves, rational cubics and rational surfaces have been formulated. Non-Uniform Rational B-spline (NURB) curves are the most widely used rational curves.

It is simply not powerful enough to use polynomials for parametric representations, because many curves (e.g., circles, ellipses, parabolas, and hyperbolas) can not be obtained this way. One way to overcome this shortcoming is to use homogeneous coordinates. For example, a curve in space is represented with four functions rather than three as follows:

\[(5.2.2.26) \quad \text{Space curve: } \mathbf{F}(u) = (x(u), y(u), z(u), w(u))\]

where \(u\) is a parameter in a closed interval \([a,b]\). Converting this curve to their conventional form yields the following:

\[(5.2.2.27) \quad \text{Space curve: } \mathbf{f}(u) = \left(\frac{x(u)}{w(u)}, \frac{y(u)}{w(u)}, \frac{z(u)}{w(u)}\right)\]

A parametric curve in homogeneous form is referred to as a rational curve. To make a distinction, we shall call a curve in polynomial form a polynomial curve.
Example. This example illustrates the power of the rational form. Consider a second degree parametric form.

\[ x = f(u) = au^2 + bu + c \]
\[ y = g(u) = pu^2 + qu + r \]

Solution. The curve described by this polynomials (both values of \( x \) and \( y \)) will go to infinity if \( u \) goes to infinity. Thus, curves such as circle and ellipse that are in finite range cannot be obtained from the above parametric form. This shows a weakness of parametric form.

Now, it will be shown that the above quadratic parametric representation really describes a conic curve. Assuming \( a \) and \( p \) are both non-zero and dividing the first and second equation by \( a \) and \( p \), respectively, leads to

\[ \frac{x}{a} = \frac{f(u)}{a} = u^2 + \frac{b}{a} u + \frac{c}{a} \]
\[ \frac{y}{p} = \frac{g(u)}{p} = u^2 + \frac{q}{p} u + \frac{r}{p} \]

Now subtracting the second from the first to eliminate the \( u^2 \) term and then solving for \( u \) yields:

\[ u = \frac{p(x - c) - a(y - c)}{bp - aq} \]
Finally, plugging this \( u \) back into the first equation (of the parametric form) gives

\[
x = a \left[ \frac{p(x-c) - a(y-c)}{bp - aq} \right]^2 + b \left[ \frac{p(x-c) - a(y-c)}{bp - aq} \right] + c
\]

Clearing the denominators and rearranging terms shows

\[
Ax^2 - 2Bxy + Cy^2 + Dx + Ey + F = 0
\]

where

\[
A = ap^2, \quad B = a^2 p, \quad C = a^3
\]

This curve is a parabola because \( B^2 - AC = 0 \).
The rational form of a circle can be found beginning with the intersection of a circle with center at the origin and radius 1 and a line from the south pole (0, -1) to a point (u, 0) on x-axis, where u is a parameter for the circle.

As u moves on the x-axis, its corresponding point moves on the circle. Any finite u has such a corresponding point. The infinite u corresponds to the south pole. The line joining the south pole and (u, 0) is \( x = uy + u \). The circle equation is \( x^2 + y^2 = 1 \). Plugging the line equation into the circle equation and solving for y yields two roots for y. One of them must be \( y = -1 \), since this line passes through the south pole. The other root is \( y = \frac{1 - u^2}{1 + u^2} \). Plugging this y into the line equation yields \( x = \frac{2u}{1 + u^2} \). Therefore, for each u, the corresponding point on the circle is given by

\[
\begin{align*}
x &= \frac{2u}{1 + u^2}, \\
y &= \frac{1 - u^2}{1 + u^2}
\end{align*}
\]
As a result, a circle has a **rational** parameterization. When $u$ moves to infinity, $x$ approaches 0 while $y$ approaches -1. In fact, this computation tells us more. This circle has a trigonometric parametric form $(\cos(t), \sin(t))$, where $t$ is in the range of 0 and $2\pi$. Therefore, a point on the circle can have two different representations, but their values are equal. That is, $(\cos(t), \sin(t)) = ((2u)/(1+u^2), (1-u^2)/(1+u^2))$ for some $u$. Hence, trigonometric functions $\cos(t)$ and $\sin(t)$ can be parameterized as follows:

\begin{align*}
(5.2.2.29) \quad \cos(t) &= \frac{2u}{1+u^2}, \quad \sin(t) = \frac{1-u^2}{1+u^2}
\end{align*}

Using a parameterization for $\cos(t)$ and $\sin(t)$ above, a rational parameterization for other conic curve such as an ellipse and a hyperbola is readily obtained.
Example. Beginning with the following form for the hyperbola in its normal form along with the parameterization above, parameterize the hyperbola and find the rational form.

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

Solution. Here, \(a\) and \(b\) are the semi-major axis and semi-minor axis lengths. It is easy to verify that the following is a correct parameterization for the hyperbola:

\[ x = a \sec(t), \quad y = b \tan(t) \]

Since \(\sec(t) = \frac{1}{\cos(t)}\) and \(\tan(t) = \frac{\sin(t)}{\cos(t)}\), plugging the rational parameterization for \(\sin(t)\) and \(\cos(t)\) into the above equations will convert them from a trigonometric parametric form to a rational one as follows:

\[(5.2.2.30) \quad x = a \frac{1 + u^2}{2u}, \quad y = b \frac{1 - u^2}{2u}, \quad 0 < |u| < \infty \]
Remember that the b-spline curve is weighted sum of its control points.

\[ P(u) = \sum_{j=0}^{n} P_j B_j^k(u), \quad 0 \leq u \leq u_{\text{max}} \]

Since the weights depend on the knot vector only, it is useful to add another weight \( w_j \) to every control point that can be set independently. The weights can be added as

\[ P(u) = \sum_{j=0}^{n} N_j^k(u) P_j, \quad N_j^k(u) = \frac{w_j B_j^k(u)}{\sum_{j=0}^{n} w_j B_j^k(u)}, \quad 0 \leq u \leq u_{\text{max}} \]

Here, \( N_j^k \) is \textbf{NURBS basis functions}. Thus, the NURBS curve of degree \( k \) is defined as weighted sum of control points with NURBS basis functions.
Some of Valuable Properties of NURBS Curves and Surfaces

• invariant under affine as well as perspective transformations (*affine invariance* and *projective invariance*).
• single mathematical form for both analytical and free-form shapes.
• flexibility to design variety of shapes.
• less memory is used when storing shapes in comparison to other methods.
• numerical algorithms can evaluate them easily and quickly.
Example. Using three points \([2,0], [3/2,1],\) and \([0,0]\) plot the second degree NURBS curve with various weights of \(w=[1,1,1], [1,0.5,1], [1,0,1],\) and \([1,-0.5,1].\)

Solution. For three control points \(n = 2\) and for the second degree \(k=2.\) The uniform knots can be computed by the program in the B-spline example. The following shows result in MathCad with modified program for \(B_j^k.\) It now uses the knot vector as an input.
Input Data

\[ \begin{align*}
  x_p & := (2 \ 1.5 \ 0)^T \\
  y_p & := (0 \ 2 \ 0)^T \\
  w & := (1 \ -0.3 \ 1)^T \\
  n & := \text{rows}(x_p) - 1 \\
  k & := 2 \\
  n & = 2 \\
  u_p & := \text{UP}(n,k) \\
  u_p^T & = (0 \ 0 \ 0 \ 1 \ 1 \ 1) \\
\end{align*} \]

\[ \begin{align*}
  x(u) & := \frac{1}{n} \sum_{j=0}^{n} w_j \cdot x_p \cdot B(j,k,n,u,u_p) \\
  y(u) & := \frac{1}{n} \sum_{j=0}^{n} w_j \cdot y_p \cdot B(j,k,n,u,u_p) \\
\end{align*} \]
Shown below are various curves that vary with different weights. Note that the last shows violation of convex hull property with negative weight for midpoint.
The cases that require manipulation of curves include:

- Displaying
- Evaluating Points on Curves
- Blending
- Segmentation
- Trimming
- Intersection
- Transformation
Example. Determine the equation of a line passing through a point $P_1$ that is parallel to an existing line between $P_3$ and $P_4$ and is trimmed by a point $P_2$.

Solution: Referring to the figure below

![Diagram of lines and points](image)

**Figure 5.3.1.** A line parallel to an existing line

the equation can be written as

\[(5.3.2) \quad P = P_1 + L\hat{n}_{3-4} \quad \text{for } 0 \leq L \leq L_2\]

where

\[
\hat{n}_{3-4} = \frac{P_4 - P_3}{|P_4 - P_3|} = \frac{(x_4 - x_3)i + (y_4 - y_3)j + (z_4 - z_3)k}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}} \quad \text{and} \quad L_2 = (P_2 - P_1) \cdot \hat{n}_{3-4}
\]
Example. Find the mathematical expression that determines the distance between a point $P_3$ and a line between $P_1$ and $P_2$.

Solution: From the inner product of two lines, the equation can be written as

![Diagram of points $P_1$, $P_2$, $P_3$, and a line $L$]

Figure 5.3.2. Distance of a point from a line

Using the relation below and noting that $|P_3 - P_1| \sin \theta = D$

(5.3.4) $| (P_2 - P_1) \times (P_3 - P_1) | = |P_2 - P_1| |P_3 - P_1| \sin \theta$

We can find $D$ from the following

(5.3.5) $|P_2 - P_1| |P_4 - P_3| \sin \theta = LD = | (P_2 - P_1) \times (P_3 - P_1) |$
Example. A closed figure with straight sides is called a polygon in engineering graphics and frequently used in designs of products. When the interior angles are same and the lengths of sides are also all same, the polygon is called a \textit{regular} polygon. Construction of a polygon is done by finding the vertices that are generally intersections of a line and an arc. For a given side (say, 100 units) of a seven-sided polygon, use the parametric representations of lines and arcs to find the vertices. Connect them and plot the polygon.
Solution: The general method to construct a regular polygon is shown below. This uses a semicircle, radial lines at equal angles, and arcs. Thus, each vertex is the intersection of a radial line and an arc. Defining an arc and a radial line parametrically, the intersection can be determined one by one. For example, the vertex 3 is the intersection of two curves

\[
\begin{align*}
  x(s) &= x_2 + r \cos s \\
  y(s) &= y_2 + r \sin s
\end{align*}
\]

and

\[
\begin{align*}
  x(t) &= \frac{x_1y_3 - x_3y_1}{(x_1 - x_3)t + y_3 - y_1} \\
  y(t) &= x(t)t
\end{align*}
\]

The first curve (e.g., circle) has a parameter \( s \) and the second (e.g., line) has a parameter \( t \). The intersections of these curves can be determined by solving \( x(s) = x(t) \) and \( y(s) = y(t) \) for \( s \) and \( t \).

Figure 5.5.1. A regular poly with seven sides
Example 1. For the given state of stresses (a) determine the principal stresses and (b) state of stresses on a plane a-a.

Example 2. For a plane motion of a bar sliding down the step, find the locus of the point C that is located at one-third length from B.
Surface model is an extension of wireframe but has advantages: less ambiguous, provide realism for display with hidden lines, mesh, and shading.

- **Surface Entities**
  - Plane surface
  - Ruled (lofted) surface (surface created by two curves being blended)
  - Surface of Revolution

![Figure 5.5.1.1. Ruled surface by two boundary curves](image1)

![Figure 5.5.2. A spatial curve about an axis of revolution and its revolved surface](image2)
- Tabulated Cylinder (surface created by a curve and a vector)
- Bezier surface: only approximates the given data points permitting only global control
- B-spline: surface that can approximate and interpolate permitting local control
- Coons Patch: used to create a surface using curves that form closed boundaries in contrast to the above surfaces that use open boundaries or set of points.
- Fillet surface: B-spline surface that blends two surfaces together
- Free-form surface: formed by free-form curves that are extensions of Bezier, B-spline, and NURB curves.

**Figure 5.5.1.3.** A spatial curve along a directrix and a t

**Figure 5.5.1.4.** A closed boundary and a Coons patch
SURFACE REPRESENTATION:

As the mathematical representation of curves, the surfaces also can be represented mathematically in two ways. The non-parametric representation of a surface can be written as

\[ P(x, y, z) = [x, y, z(x, y)] \]

The surface is described completely by knowing \( z \)-values at all points in \( x-y \) domain. The parametric representation of a surface in three dimension can be written with two parameters, \( u \) and \( v \) as

\[ P(u, v) = [x(u, v), y(u, v), z(u, v)], \quad u_{\text{min}} \leq u \leq u_{\text{max}}, \quad v_{\text{min}} \leq v \leq v_{\text{max}} \]
Very important properties of any surface are tangent vectors and a normal vector at a point on the surface that are required for tool-path generation. These can be readily calculated based on the calculus. Along the $v =$ constant curve, the slope of the surface at the point P is

\[(5.5.2.3) \quad \mathbf{p}_u = \frac{\partial \mathbf{P}}{\partial u} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}\]

and along the $u =$ constant curve,

\[(5.5.2.4) \quad \mathbf{p}_v = \frac{\partial \mathbf{P}}{\partial v} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}\]

Usually, these two vectors are presented in combination as

\[(5.5.2.5) \quad \begin{bmatrix} \mathbf{p}_u \\ \mathbf{p}_v \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}\]

The normal unit vector to a surface is another important geometric property that can be calculated by

\[(5.5.2.6) \quad \mathbf{n}(u, v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{|\mathbf{p}_u \times \mathbf{p}_v|}\]
The distance between two points on a curved surface is also an important property of surface analysis. The infinitesimal distance between two points \((u, \nu)\) and \((u + du, \nu + d\nu)\) that is given by

\[(5.5.2.7) \quad ds^2 = P_u \cdot P_u du^2 + P_u \cdot P_v du d\nu + P_v \cdot P_v d\nu^2\]

can be integrated between any two points with a parameter \(t\) as

\[(5.5.2.8) \quad S = \int_{t_1}^{t_2} \left[ P_u \cdot \frac{du}{dt} \frac{du}{dt} + P_u \cdot P_v \frac{du}{dt} \frac{d\nu}{dt} + P_v \cdot P_v \frac{d\nu}{dt} \frac{d\nu}{dt} \right] dt\]
Example 5.5.2.1. Non-parametric form of a sphere \((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\) can be expressed in parametric form as \(P(u,v) = [x_0 + r \cos u \cos v, y_0 + r \cos u \sin v, z_0 + r \sin u]\), \(-\pi/2 \leq u \leq \pi/2, 0 \leq v \leq 2\pi\). This form can be used to find tangent vectors at a certain point as well as distance along a path on the surface.
Plane Surface: $P(u,v) = P_0 + u(P_1-P_0) + v(P_2-P_0)$
Ruled Surface
Surface of Revolution
Tabulated Cylinder
A plane surface that passes through three points, \( P_0, P_1, \) and \( P_2 \), is given by

\[
P(u, \nu) = P_0 + u(P_1 - P_0) + \nu(P_2 - P_0), \quad 0 \leq u \leq 1, \quad 0 \leq \nu \leq 1
\]

The surface normal unit vector then is

\[
\mathbf{n}(u, \nu) = \frac{(P_1 - P_0) \times (P_2 - P_0)}{|(P_1 - P_0) \times (P_2 - P_0)|}, \quad 0 \leq u \leq 1, \quad 0 \leq \nu \leq 1
\]

Once the normal unit vector is known, the surface can be also expressed in nonparametric form as

\[
(P - P_0) \cdot \mathbf{n} = 0
\]
Bilinear surface is a linear interpolation of four corners in two different directions \((u, v)\).

\[
P_{u,0}(u) = (1-u)P_{0,0} + uP_{1,0} \\
P_{u,1}(u) = (1-u)P_{0,1} + uP_{1,1} \\
P(u, v) = (1-v)P_{u,0} + vP_{u,1}
\]
A ruled surface is generated by joining two space curves (i.e., rails) with a straight line (i.e., ruling or generator). If the two curves are denoted by $F(u)$ and $G(u)$, respectively, for a value of $u$, then the parametric equation is given by

$$\mathbf{P}(u, v) = (1 - v)G(u) + vF(u), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1$$
A revolved surface is generated by a space curve about an axis of rotation. The parametric representation of the curve in the working coordinate system that has x and y-axes on the perpendicular plane to the axis of rotation that is z-axis can be expressed by

\[
P(\mu, \nu) = r_2(\mu) \cos \nu \mathbf{n}_x + r_2(\mu) \sin \nu \mathbf{n}_y + z(\mu) \mathbf{n}_z, \quad 0 \leq \mu \leq 1, \quad 0 \leq \nu \leq 2\pi
\]
A tabulated cylinder is generated by moving a straight line (i.e., generatrix) along a space curve (i.e., directrix). The generatrix is always parallel to a fixed vector. The parametric representation of the curve can be expressed by

\[ P(u, v) = F(u) + v n \cdot, \quad 0 \leq u \leq u_{\text{max}}, \quad 0 \leq v \leq v_{\text{max}} \]

where \( n \) is the unit vector in the direction of generatrix.
PARAMETRIC REPRESENTATION OF SYNTHETIC SURFACES
1. Hermite Bicubic Surface

This surface is formed by Hermite cubic splines running in two different directions. It interpolates to a finite number of data points to form the surface. The bicubic interpolation is an invaluable tool used in image processing.

\[ P(u, v) = \sum_{i=0}^{3} \sum_{j=0}^{3} C_{ij} u^i v^j, \quad 0 \leq u, v \leq 1 \]

In a matrix form it can be expressed

\[ P(u, v) = U^T [C] V = U^T [M_H] [B] [M_H]^T V \]

where \( U^T = \{u^3, u^2, u, 1\} \), \( V = \{v^3, v^2, v, 1\}^T \) and the connectivity matrix \([B]\) has \(4 \times 4 = 16\) coefficients. Referring to Figure 5.5.2.1 and applying the boundary conditions (i.e., continuity and tangency) at data points determines all coefficients. Here

\[ [B] = \begin{bmatrix}
    P_{00} & P_{01} & \frac{\partial P}{\partial v}_{00} & \frac{\partial P}{\partial v}_{01} \\
    P_{10} & P_{11} & \frac{\partial P}{\partial v}_{10} & \frac{\partial P}{\partial v}_{11} \\
    \frac{\partial P}{\partial u}_{00} & \frac{\partial P}{\partial u}_{01} & \frac{\partial^2 P}{\partial u \partial v}_{00} & \frac{\partial^2 P}{\partial u \partial v}_{01} \\
    \frac{\partial P}{\partial u}_{10} & \frac{\partial P}{\partial u}_{11} & \frac{\partial^2 P}{\partial u \partial v}_{10} & \frac{\partial^2 P}{\partial u \partial v}_{11}
\end{bmatrix} \]

This matrix can be determined by imposing the smoothness conditions at data points joining two adjacent panels.
Example 5.5.4.1. Find the bicubic spline surface for four corners, \((-2,-3,5), (4,-5,1), (2,6,6),\) and \((-1,5,4)\) and evaluate the point at \(u = 1/3\) and \(v = 1/2\). Assume that all the slopes are zeros at corners.

Solution: The \(B\) matrix for the \(z\)-coordinates is given below with the surface plot.

\[
B_z = \begin{bmatrix}
5 & 4 & 0 & 0 \\
1 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The point at \(u = 1/3\) and \(v = 1/2\) is found \(P(1/3,1/2) = (-1/3,47/54,229/54)\).
Example 5.5.4.2. Find the bicubic spline surface for four corners, \((-2,-3,5), (4,-5,1), (2,6,6),\) and \((-1,5,4)\) and evaluate the point at \(u = 1/3\) and \(v = 1/2\). Assume that the boundaries are straight lines.

Solution: The \(B\) matrix for the \(z\)-coordinates is given below with the surface plot.

\[
\mathbf{B}_z = \begin{bmatrix}
5 & 4 & -1 & -1 \\
1 & 6 & 5 & 5 \\
-4 & 2 & 0 & 0 \\
-4 & 2 & 0 & 0
\end{bmatrix}
\]

Here, more equations were obtained by the straight lines connecting corners. For example,

\[
\mathbf{L}(u,0) \equiv \mathbf{L}_{01} = \mathbf{P}_{00} + u(\mathbf{P}_{10} - \mathbf{P}_{00}), \quad 0 \leq u \leq 1
\]

\[(5.5.4.4)\]

\[
\frac{\partial \mathbf{P}}{\partial u} \bigg|_{00} = \frac{\partial \mathbf{L}(u,0)}{\partial u} \bigg|_{00} = (\mathbf{P}_{10} - \mathbf{P}_{00})
\]

The point at \(u = 1/3\) and \(v = 1/2\) is found \(\mathbf{P}(1/3,1/2) = (0.5/6, 25/6)\).
Example 5.5.4.3. Find the bicubic spline surface for two surfaces formed by six corners, \((-2,-3,5), (4,-5,1), (10,-4,3), (-1,5,4), (2,6,6), \) and \((9,7,8)\).

Solution: There are two surfaces to be connected smoothly. The smoothness conditions can be obtained by equating the second derivatives of both surfaces at knots for twelve unknowns as

\[
\begin{align*}
\frac{\partial^2 P_{S1}}{\partial u^2} (1,0) &= \frac{\partial^2 P_{S2}}{\partial u^2} (0,0), & \frac{\partial^2 P_{S1}}{\partial v^2} (1,0) &= \frac{\partial^2 P_{S2}}{\partial v^2} (0,0), & \frac{\partial^2 P_{S1}}{\partial u \partial v} (1,0) &= \frac{\partial^2 P_{S2}}{\partial u \partial v} (0,0) \\
\frac{\partial^2 P_{S1}}{\partial u^2} (1,1) &= \frac{\partial^2 P_{S2}}{\partial u^2} (0,1), & \frac{\partial^2 P_{S1}}{\partial v^2} (1,1) &= \frac{\partial^2 P_{S2}}{\partial v^2} (0,1), & \frac{\partial^2 P_{S1}}{\partial u \partial v} (1,1) &= \frac{\partial^2 P_{S2}}{\partial u \partial v} (0,1)
\end{align*}
\]

(5.5.4.5)

Out of twelve unknowns in \([B]\) matrix above only two values are uniquely obtained for each direction as

\[
\begin{align*}
\left. \frac{\partial P_{S1}}{\partial u} \right|_{10} &= \begin{bmatrix} 9, \ -3, \ -\frac{3}{4}, \ -3, \ -2 \end{bmatrix}^T, & \left. \frac{\partial P_{S1}}{\partial u} \right|_{11} &= \begin{bmatrix} 15, \ 3, \ 3, \ 2, \ 2 \end{bmatrix}^T
\end{align*}
\]

Thus, the rest of values are set to zeros and the two surfaces are plotted below.
Bilinear surface is a linear interpolation of four corners in two different directions \((u, v)\).

\[
P_{u,0}(u) = (1-u)P_{0,0} + uP_{1,0}
\]
\[
P_{u,1}(u) = (1-u)P_{0,1} + uP_{1,1}
\]
\[
P(u, v) = (1-v)P_{u,0} + vP_{u,1}
\]
Bilinear surface generated by four corners has straight sides and produce quite flat surfaces. In contrast, Coon’s surface uses four side curves.

\[
P_{LR}(u, v) = (1 - u)P_L(v) + uP_R(v)
\]

\[
P_{BT}(u, v) = (1 - v)P_B(u) + vP_T(u)
\]

\[
P(u, v) = P_{LR}(u, v) + P_{BT}(u, v)
\]

The surface obtained as above does not produce the end curves. Thus, evaluating the surface on boundaries and forcing it to confirm to the boundary curves yields extra-terms that must be subtracted.
Evaluating the surface along edges one finds extra terms that must be subtraced.

\[
\begin{align*}
\mathbf{P}_{LR}(u, v) &= (1-u)\mathbf{P}_L(v) + u\mathbf{P}_R(v) \\
\mathbf{P}_{BT}(u, v) &= (1-v)\mathbf{P}_B(u) + v\mathbf{P}_T(u) \\
\mathbf{P}(0, v) &= \mathbf{P}_{LR}(0, v) + \mathbf{P}_{BT}(0, v) \\
&= \mathbf{P}_L(v) + (1-v)\mathbf{P}_B(0) + v\mathbf{P}_T(0) \\
&= \mathbf{P}_L(v) + (1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0,1} \quad \text{(must)} \\
&= \mathbf{P}_L(v)
\end{align*}
\]

On the left edge, the extra terms are identified as

\[
(1-v)\mathbf{P}_{0,0} + v\mathbf{P}_{0,1}
\]
Similarly, the other extra terms are found along the other edges and all surplus terms are

\[ \text{surplus} = (1 - v)P_{0,0} + vP_{0,1} \]
\[ + (1 - v)P_{1,0} + vP_{1,1} \]
\[ + (1 - u)P_{0,0} + uP_{1,0} \]
\[ + (1 - u)P_{0,1} + uP_{1,1} \]

\[ P(u, v) = P_{LR}(u, v) + P_{BT}(u, v) - \text{surplus} \]

- **Advantages:**
  - Follows boundary curves
- **Limitation:**
  - Not able to control internal shape
For given data along four edges, use cubic Bezier curves along the edges to create Coon’s patch.

\[
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0, 0, 0, 0 \\
0, -1, 1/2, 0 \\
0, 1, 2, 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0, 0, 0, 0 \\
0, -1, 1/2, 0 \\
0, 1, 2, 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0, 0, 0, 0 \\
0, -1, 1/2, 0 \\
0, 1, 2, 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0, 0, 0, 0 \\
0, -1, 1/2, 0 \\
0, 1, 2, 0
\end{bmatrix}
\]
First find the curves along the edges. Their three components in (u,v) are

\[
\begin{align*}
\text{cvL} &= \begin{bmatrix}
0 \\ 3(1-u)^2 u + 6(1-u)u^2 + 4u^3 \\ 3(1-u)^2 u - 3(1-u)u^2
\end{bmatrix} \\
\text{cvB} &= \begin{bmatrix}
3(1-v)^2 v + 6(1-v)v^2 + 3v^3 \\ 0 \\ -3(1-v)^2 v + \frac{3(1-v)v^2}{2}
\end{bmatrix} \\
\text{cvR} &= \begin{bmatrix}
3(1-u)^3 + 9(1-u)^2 u + 9(1-u)u^2 + 3u^3 \\ 3(1-u)^2 u + 6(1-u)u^2 + 4u^3 \\ 3(1-u)^2 u + 6(1-u)u^2
\end{bmatrix} \\
\text{cvT} &= \begin{bmatrix}
3(1-v)^2 v + 6(1-v)v^2 + 3v^3 \\ 4(1-v)^3 + 12(1-v)^2 v + 12(1-v)v^2 + 4v^3 \\ 6(1-v)^2 v + \frac{3(1-v)v^2}{2}
\end{bmatrix}
\end{align*}
\]

L, R, B, and T refer to left, right, bottom, and top.
The plot is shown below.

L, R, B, and T refer to left, right, bottom, and top.
The Coon’s patch is then obtained as below and the plot is shown with data points.
2. Bezier Surface

Bezier surface is an extension of Bezier curve and interpolates to a finite number of data points. It can be expressed as

\[
P(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} B_{i,m}(u) B_{j,n}(v), \quad 0 \leq u, v \leq 1
\]
Example 5.5.4.4. Find the Bezier surface for four corners, \((-2,-3,5), (4,-5,1), (2,6,6),\) and \((-1,5,4)\).

Solution: For \(x(u,v) = P_{i,j} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix},\) for \(y(u,v) = P_{i,j} = \begin{bmatrix} -3 & -5 \\ 5 & 6 \end{bmatrix},\) and for \(z(u,v) = P_{i,j} = \begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix}.\) The parametric surface plot is same as the bicubic spline surface with boundaries of straight lines above.
Example 5.5.4.5. Find the Bezier surface for six corners, $(-2,-3,5)$, $(4,-5,1)$, $(10,-4,3)$, $(-1,5,4)$, $(2,6,6)$, and $(9,7,8)$.

Solution: With more data points the surface is shown with circled data points.
Example [Programming]

Bezizer surface

BiBezizer(p, u, v) :=

\[\begin{aligned}
m &\leftarrow \text{rows}(p) - 1 \\
n &\leftarrow \text{cols}(p) - 1 \\
n &\leftarrow 0 \\
\text{for } i \in 0..m &
\text{for } j \in 0..n \\
& \quad \text{s} \leftarrow s + p_{i,j} \cdot \frac{m!}{i!(m-i)!} \cdot u^i \cdot (1-u)^{m-i} \cdot \frac{n!}{j!(n-j)!} \cdot v^j \cdot (1-v)^{n-j} \\
\end{aligned}\]
collect $x,y,z$-data for parametric surface plot

\[
i := 0 \ldots 10 \quad j := 0 \ldots 20
\]

\[
 xd_{i,j} := \text{BiBezier}(xp, \frac{1}{10} \cdot i, \frac{1}{20} \cdot j) \quad yd_{i,j} := \text{BiBezier}(yp, \frac{1}{10} \cdot i, \frac{1}{20} \cdot j) \quad zd_{i,j} := \text{BiBezier}(zp, \frac{1}{10} \cdot i, \frac{1}{20} \cdot j)
\]

plotting

$(xd, yd, zd)$, $(xp, yp, zp)$
3. B-Spline Surface

B-Spline surface interpolates to a finite number of data points.

\[(5.5.4.7) \quad P(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i-p}^{P}(u) B_{j-q}^{Q}(v), \quad 0 \leq u \leq u_{\text{max}}, \quad 0 \leq v \leq v_{\text{max}}\]
Example [Programming]

\[
\begin{bmatrix}
1 & 3 & 4 & 6 & 9 \\
1 & 3 & 4 & 6 & 9 \\
1 & 3 & 4 & 6 & 9 \\
1 & 3 & 4 & 6 & 9 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 \\
9 & 9 & 9 & 9 & 9 \\
\end{bmatrix}
\begin{bmatrix}
7 & 3 & 4 & 6 & 9 \\
5 & 3 & 9 & 6 & 3 \\
4 & 5 & 2 & 5 & 4 \\
5 & 9 & 4 & 3 & 6 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 5 & 25 & 6 & 2 \\
4 & 5 & 8 & 6 & 3 \\
5 & 6 & 8 & 6 & 4 \\
\end{bmatrix}
\]

degrees \quad p := 3 \quad q := 3

number of data points \quad m := \text{rows}(xp) - 1 \quad n := \text{cols}(xp) - 1

\begin{align*}
ux & := \text{UP}(m, p, 0) \\
uy & := \text{UP}(n, q, 0)
\end{align*}

Range of \( u, v \) \quad u := 0, 0.05..m - p + 1 \quad v := 0, 0.05..n - q + 1

\begin{align*}
\text{umax} & := m - p + 1 \\
\text{vmax} & := n - q + 1
\end{align*}
bi-B-spline

\[ x(u,v) := \sum_{i=0}^{m} \sum_{j=0}^{n} x_{p_{i,j}} B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \]

\[ y(u,v) := \sum_{i=0}^{m} \sum_{j=0}^{n} y_{p_{i,j}} B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \]

\[ z(u,v) := \sum_{i=0}^{m} \sum_{j=0}^{n} z_{p_{i,j}} B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \]
collect $x,y,z$-data for parametric surface plot

$$i := 0 .. 10 \quad j := 0 .. 20$$

$$x_d_{1,j} := x \left( \frac{umax}{10} \cdot i, \frac{vmax}{20} \cdot j \right) \quad y_d_{1,j} := y \left( \frac{umax}{10} \cdot i, \frac{vmax}{20} \cdot j \right) \quad z_d_{1,j} := z \left( \frac{umax}{10} \cdot i, \frac{vmax}{20} \cdot j \right)$$

plotting

$$(x_d, y_d, z_d), (x_p, y_p, z_p), (x_p, y_p, z_p)$$
The curve to find is

\[ P(u) := \frac{\sum_{j=0}^{n} w_j \cdot P_j \cdot (B_j)^k(u)}{\sum_{j=0}^{n} w_j \cdot (B_j)^k(u)} \]
\[
\begin{align*}
\text{bi-nurbs} & \\
\text{denom}(u,v) & := \sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} \cdot B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \\
x(u,v) & := \sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} \cdot x_{i,j} \cdot B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \\
& \quad \frac{\text{denom}(u,v)}{} \\
y(u,v) & := \sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} \cdot y_{i,j} \cdot B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \\
& \quad \frac{\text{denom}(u,v)}{} \\
z(u,v) & := \sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} \cdot z_{i,j} \cdot B(i,p,m,u,ux) \cdot B(j,q,n,v,uy) \\
& \quad \frac{\text{denom}(u,v)}{} \\
y(0.9,1.2) & = 7.50316
\end{align*}
\]
collect x,y,z-data for parametric surface plot

\[i := 0..10\quad j := 0..20\]

\[\begin{align*}
    x_{d1,j} &= x\left(\frac{u_{max}}{10} \cdot i, \frac{v_{max}}{20} \cdot j\right) \\
    y_{d1,j} &= y\left(\frac{u_{max}}{10} \cdot i, \frac{v_{max}}{20} \cdot j\right) \\
    z_{d1,j} &= z\left(\frac{u_{max}}{10} \cdot i, \frac{v_{max}}{20} \cdot j\right)
\end{align*}\]

plotting

\[(x_d, y_d, z_d),(x_p, y_p, z_p),(x_p, y_p, z_p)\]
In addition to the surfaces mentioned above, other surfaces are blending surface, offset surface, triangular patches, sculptured (or free-form) surface that is a collection of interconnected and bounded parametric patches together with blending and interpolation formulas, and rational parametric surface.
Surface Manipulations

As in the curve manipulations, the surfaces have to be manipulated for various reasons in the CAD designs. Some of the applications are

1. Displaying: mesh, shading, and colors. For nested surfaces, x-ray and transparency techniques.

Figure 5.5.5.1. Various surface display
2. Evaluating points and curves on surfaces

![Diagram showing points and curves on surfaces]

**Figure 5.5.5.2.** (a) Distance between two points and (b) curvature of a curve

3. Intersection – Surfaces often intersect in design and their mathematical representation is required in CAD.
4. Segmentation – Splitting a surface along a curve such as intersection of two surfaces is very common in design a model and needs mathematical manipulation. In the above figure, the cylindrical surface intersects with the spherical surface.
5. Trimming – Either part of a split surface is often removed in design. In the above figure, the split cylindrical surface inside the spherical surface has been removed.
6. Projection – This is another very important technique for displaying the 3D model in CAD using various views.
7. Transformation – For all the display views, the mathematical representation of the model must be manipulated from that in the database through various transformations such as scaling and rotation.
5.5-6 Design and Engineering Applications

There are numerous applications of surfaces in design. The following shows some of problems for illustration.

Example 5.5.6.1. Construct a surface for a given set of data points.

Example 5.5.6.2. Construct a surface for a given set of data points with revolution two intersecting pipes with an angle for welding.

Example 5.5.6.3. A 4 in. air-conditioning duct to 4 in. elbow with radius 8 in. to 4 in. and 6 in. diameter duct to 12 in. long and 10x10 in square duct.

Example 5.5.6.4. Construct a surface of revolution for a given set of points.

Example 5.5.6.5. Two pipes with diameter D whose axes form an angle q are to be joined for welding. Find the perimeter for welding.
Solid Model is based on informationally complete (or spatial addressability), valid, and unambiguous representation of objects and stores geometric data as well as topological information of associated objects. This representation permits automation and integration of tasks such as interference analysis, mass property calculation, finite element modeling, CAPP (computer-aided process planning), machine vision, and NC machining.

It is very easy to define an object with a solid model than other two previous modeling techniques (curves and surfaces) because solid models do not need individual locations as with wireframe models.
The above figure illustrates the difference between geometry and topology. The geometry that defines the object is the lengths of lines, areas of surfaces, the angles between the lines, and the radius and the center of the cylinder and the height. On the other hand, topology (sometimes called combinatorial structure), is the connectivity and associativity of the object entities. It has to do with the notion of neighborhood and determines the relational information between object entities. From a user point of view, geometry is visible and topology is considered to be nongraphical relational information that is stored in solid model databases and are not visible to users.
There are various basic building blocks, so called, primitives that can be combined in certain boolean operations to construct complex models. They include:

- block
- cylinder
- cone
- sphere
- wedge
- Torus

In the previous figure, a block and a cylinder were combined with union (i.e., addition) and difference (i.e., subtraction). One more available Boolean operation is intersection.
Example 5.6.1.1. Explain how to construct the solid model of the bearing support with primitives and boolean operations.
Underlying fundamentals of solid modeling theory are geometry, topology, geometric closure, set theory, regularization of set operations, set membership classification, and neighborhood. Solid representation is based on the notion that a physical object divides an n-dimensional space, $E_n$, into two regions: interior and exterior separated by the boundaries. A region is a portion of space $E_n$ and the boundary of a region is a closed surface.

The set in the set theory is a collection of objects and the operations include complement, union, and intersection. Further, regularized set operations are used to avoid irregular object created by boolean operations. Regular set is a geometrically closed set. The set membership classification determines if some objects intersect with a given object.
Half-spaces are unbounded geometric entities; each one of them divides the representation space into two infinite portions, one filled with material and the other empty. By combining half-spaces in a building block fashion, various solids can be constructed.

Thus, a solid model of an object can be defined as a point set \( S \) in three-dimensional Euclidean space \( E^3 \). If three sets for a region is denoted by \( S_i \) (interior set), \( S_e \) (exterior set), and \( S_b \) (boundary set), then the set for the solid model can be expressed as

\[
S_{SM} = S_i \cup S_b
\]
The mathematical properties that the solid model should capture can be stated as:

- Rigidity
- Homogeneous three-dimensionality (ie. no dangling boundaries)
- Finiteness and finite describability
- Closure under rigid motion and regularized boolean operations
- Boundary determinism (ie. Boundary must contain the solid)

The mathematical implication of the above properties suggests that valid solid models are bounded, closed, regular, and semi-analytic subsets of $E^3$. These subsets are called r-sets (ie. regularized sets), which are curved polyhedra with well-behaved boundaries. Here, “regular” means no dangling portion and “semi-analytic” means no oscillation in value.
Various solid representation schemes require several underlying fundamentals of solid modeling theory. They include geometry, topology, geometric closure, set theory, regularization of set operations, set membership classification, and neighborhood. For detailed discussion of these topics, the readers are encouraged to refer to advanced topics on solid modeling.

In the following some of most popular solid modeling techniques are discussed.
Boundary Representation (B-rep): A B-rep model is one of the two most popular and widely used schemes and is based on the topological notion that a physical object is bounded by a set of faces. The following shows various faceted B-rep solids.
Constructive Solid Geometry (CSG): This is based on the topological notion that a physical object can be divided into a set of primitives that can be combined in a certain order following a set of rules (i.e. Boolean operations) to form the object. The basic elements are block, cylinder, cone, sphere, wedge, and torus and building operations are Boolean operations. The following bearing support was constructed with various primitives in a certain sequence by CSG technique.
**Sweep Representation:** This is especially useful for two-and-half dimensional objects used most frequently for extruded solids and revolved solids. This is based on sweeping of a section along a path that may be linear, nonlinear, and hybrid operations. When the path is straight, it’s a simple extrusion and when it is axisymmetric, it becomes the revolution. For nonlinear path, the sweeping is done along a nonlinear curve in space. Cutting tool path simulation is one good applications of this technique.

The section may vary along the sweeping path. The following shows such an example with variable section.
**Analytic Solid Modeling (ASM):** Historically it is closely related to three dimensional isoparametric formulation of finite element analysis for 8- to 20-node hexahedral elements. This arose from the need to model complex objects for finite element analysis. ASM uses the parametric representation of an object in three-dimensional space that is a mapping of a cubical parametric domain (so called, master domain) into a solid described by the global coordinates (MCS).
Other Representations:

- primitive instancing: is based on notion of families of objects or family of parts.
- cell decomposition scheme: an object can be represented as the sum of cells.
- spatial enumeration scheme: a solid is represented by the sum of spatial cells that it occupies. The cells have fixed size.
- octree encoding scheme: is a generalization of spatial enumeration scheme with variable cell size.