Designing Conservative Sensor Detection Systems with Emitter Location Uncertainty

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Abstract—The design of sensor systems to detect an emitter at a random location with an unknown distribution is difficult because measurements are conditionally dependent and the hypothesis test is composite. This paper shows that, when sensors are at deterministic locations, these problems can be circumvented and a conservative design is achieved by adopting a least favorable distribution for the emitter location. An algorithm to achieve the design and the application to GLRT detectors are presented.

Index Terms—Sensor systems, conditional dependence, composite hypothesis, least favorable distributions, GLRT.

I. INTRODUCTION

Consider a sensor system to detect the presence of a signal emitter at a random location within a region of interest. Sensors at known fixed locations collect and process multiple measurements, and send the result to a fusion center. The fusion center combines the received statistics to decide between hypotheses $H_0$ (no signal emitted) and $H_1$ (signal emitted), as illustrated in Fig. 1. Sensor detection systems have many applications; for instance, sensors can detect the unauthorized release of low-power radiation sources in a metropolitan area [1], [2] or detect submarines in a region of the ocean [3].

When designing such a sensor detection system, the designer faces two issues: the conditional dependence among sensor measurements and the composite hypothesis $H_1$.

In general, sensor measurements are conditionally dependent when they depend on the distance between emitter and sensors. In this case, the analysis becomes difficult because of the complicated form of optimal detectors and the difficulty in obtaining correlation parameters [2], [4], [5] [6, p. 110].

The hypothesis $H_1$ is composite because, in many applications, the distribution for the emitter location is unknown. Since the measurements depend on the emitter location and there are uncountably many possible distributions for it, $H_1$ is composite [7, p. 191]. Designs under composite hypothesis are difficult because uniformly most powerful detectors may not exist. Even with non-optimum systems, it is difficult to ensure a detection performance for all distributions in $H_1$.

One way to avoid the problem caused by the composite hypothesis $H_1$ is to assume a least favorable distribution (LFD) for the emitter location. When such a distribution is adopted, $H_1$ becomes simple and the designer can use the well developed techniques for simple hypotheses tests. This approach was taken by [8], which showed that adopting an LFD can solve not only the problem of composite hypothesis, but also the problem of conditional dependence in that the measurements become conditionally i.i.d. However, the results of [8] require that sensor locations be random variables and do not apply when sensor locations are instead deterministic.

This paper shows that the approach of assuming an LFD for the emitter location can also be applied and the issues of conditional dependence and composite hypothesis can be circumvented when sensors are located at known deterministic locations. Furthermore, as discussed in Section III, this approach produces a conservative design: a system that satisfies a prescribed detection requirement under the LFD will also satisfy the requirement under the actual unknown emitter location distribution. This paper builds upon and expands the preliminary work by this author in [9], which shows that finding an LFD for the emitter location can be equivalent to solving a problem from the field of operations research. Although this equivalence is fruitful, it does not hold for all systems of interest, such as the distributed detection system in which the fusion center decides based on the majority rule.

To apply the approach of assuming an LFD for the emitter location to a broad class of systems, this paper proposes the use of the numerical branch and bound minimization method proposed in [10]. Although this method is well defined in [10], it requires the definition of tight upper and lower bounds for the minimizing function, which are not defined in [10] but are defined in the Section IV of this paper.

Although most of the paper considers the detection of a single emitter and assumes that its location is random, the approach of LFDs can be used for multiple emitters as shown in Section VI, which also presents conditions where a single emitter is the worst case for the detection problem; and the case of an emitter at an unknown deterministic location is considered in Section III-C.

Lastly, Section VII shows that the approach of assuming an LFD for the emitter location can also be used to design conservative systems based on the Generalized Likelihood Ratio Test (GLRT), which is a common approach taken when the emitter location is unknown [1].

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II. MODEL DESCRIPTION AND DESIGN OBJECTIVE

Consider a sensor detection system with \( K \) sensors at fixed known deterministic locations \( \{ l_i \}_{i=1}^K \) in \( \mathbb{R}^2 \).

Given a region \( S_e \subset \mathbb{R}^2 \), assumed to be compact and with non zero area, the goal of the system is to decide whether \( S_e \) does not contain a signal emitter (hypothesis \( H_0 \)) or \( S_e \) contains a single signal emitter at location \( L_e \) (hypothesis \( H_1 \)). Section VI discusses the multiple emitters case.

The emitter location \( L_e \) is assumed to be a random variable on \( S_e \) with an unknown distribution \( P_{L_e} \), where \( P_X \) represents the probability measure induced by a random variable \( X \). A realization of \( L_e \) is represented by \( l_e \). Section III-C discusses the case of an unknown deterministic emitter location.

To detect the emitter, each sensor \( i \) collects a batch of \( M \) measurements, \( Z_i := (Z_{i,1}, \ldots, Z_{i,M}) \), with \( Z_{i,j} \) having a distribution that depends on the distance \( \| l_i - l_e \| \) between the sensor and the emitter through a decay function \( \xi \), which is assumed to be non-zero, real, bounded, positive, nonincreasing, and continuous. During the collection of the \( M \) measurements, the emitter is assumed to be stationary.

To define the distributions of \( Z_{i,j} \) under \( H_0 \) and \( H_1 \), let \( \{F(z|x) : x \geq 0\} \) be a family of cumulative distribution functions (c.d.f.) parametrized by \( x \), where \( F(z|x) \) is assumed known. Below are some examples of interest:

- Gaussian case: \( F(z|x) \) is the c.d.f. of a Gaussian random variable with variance \( \sigma^2 > 0 \) and mean \( x \);
- Poisson case: \( F(z|x) \) is the c.d.f. of a Poisson random variable with parameter \( \lambda + x \) for some \( \lambda > 0 \).

The conditional distributions of \( Z_{i,j} \) are defined by:

\[
P_0[Z_{i,j} \leq z] = F(z|0),
\]

\[
P_1[Z_{i,j} \leq z] = \int_{S_e} F(z|\|(l_i - l_e)\|) \, dP_{L_e}(l_e),
\]

where \( P_0 \) and \( P_1 \) respectively define the underlying probability measures under \( H_0 \) and \( H_1 \); and \( \int f(x) \, dP_X(x) \) represents Lebesgue integral of \( f \) under the probability measure \( P_X \).

Under \( H_0 \), \( \{Z_{i,j}\}_{i=1}^K \) are conditionally independent and identically distributed (i.i.d.). Under \( H_1 \), \( \{Z_{i,j}\}_{i=1}^K \) are conditionally dependent in general due to the dependence on the distances between \( \{l_i\}_{i=1}^K \) and a common \( L_e \); however, \( \{Z_{i,j}\}_{i=1}^K \) are assumed conditionally independent given \( L_e = l_e \), i.e., for any \( \{z_{i,j}\}_{i=1}^K \),

\[
P_1 \left[ \bigcap_{i=1}^K \left( \bigcap_{j=1}^M \{Z_{i,j} \leq z_{i,j}\} \right) | L_e = l_e \right] = \prod_{i=1}^K \prod_{j=1}^M P_1[Z_{i,j} \leq z_{i,j} | L_e = l_e] = \prod_{i=1}^K \prod_{j=1}^M F(z_{i,j} | \|(l_i - l_e)\|),
\]

where it is noted that each of the \( M \) measurements collected by the sensor \( i \) depends on the same \( \| l_i - l_e \| \).

Each sensor \( i \) processes \( Z_i \), with a sensor function \( \phi_i \), whose output is transmitted to the fusion center through a dedicated error-free communication channel. The fusion center uses the outputs of \( \phi_1, \ldots, \phi_K \) as the input to a fusion function \( \phi_D \) that decides between \( H_0 \) and \( H_1 \). The set \( \phi = \{\phi_0, \phi_1, \ldots, \phi_K\} \) represents the detection system. When \( \phi_i(Z_i) = Z_i \) for \( i \neq 0 \), \( \phi \) is a centralized detection system; and when \( \phi_i \) outputs \( \overline{U}_i \in \{0, \ldots, U_{\max}\} \) for some integer \( U_{\max} > 0 \) and \( i \neq 0 \), \( \phi \) is a distributed detection system.

For a given \( \phi \) with sensors at \( \{l_i\}_{i=1}^K \), the objective is to determine the minimum number of measurements \( M \) necessary for \( \phi \) to satisfy a minimum probability of detection \( (\beta_{\min}) \) for a maximum probability of false alarm \( (\alpha_{\max}) \). Let \( \alpha(\phi) \) be the probability of false alarm obtained by \( \phi \); and let \( \beta(P_{L_e}, \phi) \) be the probability of detection obtained by \( \phi \) when \( L_e \) is distributed according to \( P_{L_e} \). To facilitate the notation, \( \beta(l_e, \phi) \) is used when \( P_{L_e} \) satisfies \( P[L_e = l_e] = 1 \).

III. LEAST FAVORABLE DISTRIBUTIONS

Since \( F(z|x) \) is known and \( Z_{i,j} \) under \( H_0 \) is defined by \( F(z|0) \), \( H_0 \) is a simple hypothesis; however, since \( Z_{i,j} \) depends on \( \|l_i - l_e\| \) with \( L_e \) being a random variable with unknown distribution, the uncountably many distributions for \( L_e \) induce uncountably many distributions for \( Z_{i,j} \), which means that \( H_1 \) is a composite hypothesis [7, p. 191].

To avoid the difficulties caused by composite hypotheses, the designer could adopt the uniform distribution for \( L_e \); however, a system \( \phi \) designed under the uniform distribution may fail to satisfy the detection performance for the unknown distribution \( P_{L_e} \). Instead, to ensure that \( \phi \) satisfies the detection performance for any \( P_{L_e} \), this paper proposes that the designer adopt a least favorable distribution (LFD) for \( \phi \).

Definition: let \( \phi \) satisfy \( \alpha(\phi) \leq \alpha_{\max} \); \( P_{L_e} \) is an LFD for \( \phi \) if

\[
\forall P_{L_e}' \neq P_{L_e}, \beta(P_{L_e}', \phi) \geq \beta(P_{L_e}, \phi),
\]

which means that if \( \beta(P_{L_e}, \phi) \geq \beta_{\min} \), then \( \beta(P_{L_e}', \phi) \geq \beta_{\min} \) for the unknown \( P_{L_e} \); i.e., the design is conservative. Furthermore, by adopting \( P_{L_e} \), \( H_1 \) becomes simple and the designer no longer needs to verify that \( \phi \) satisfies the performance requirement for all possible distributions in \( H_1 \); instead, the designer only needs to verify that \( \beta(P_{L_e}, \phi) \geq \beta_{\min} \) to ensure that the requirement is satisfied for all distributions.

Note that more than one distribution may satisfy (3) and be an LFD for \( \phi \). Note further that the definition (3) is tied to the given \( \phi \); and different systems may have different LFDs and different lower bounds for the probability of detection.

A. Solving the Conditional Dependence Problem

Adopting an LFD for \( \phi \) may also solve the conditional dependence problem.

If there exists a point \( l_e^- \in S_e \) that satisfies

\[
\forall l_e \in S_e, \beta(l_e, \phi) \geq \beta(l_e^-, \phi), \quad (4)
\]

then for any \( P_{L_e}' \),

\[
\beta(P_{L_e}', \phi) \geq \int_{S_e} \beta(l_e^-, \phi) \, dP_{L_e}'(l_e), \quad (5)
\]

\[
\geq \int_{S_e} \beta(l_e^-, \phi) \, dP_{L_e}(l_e) = \beta(l_e^-, \phi), \quad (6)
\]

which means that the distribution that places the emitter at \( l_e^- \) with probability 1 satisfies (3) and is an LFD for \( \phi \).

If \( l_e^- \) that satisfies (4) exists and \( P_{L_e} \) is assumed to satisfy \( P[L_e = l_e^-] = 1 \), then, from (2), \( \{Z_{i,j}\}_{i=1}^K \) become conditionally independent under \( H_1 \).
B. Existence Conditions

How to determine whether \( l_e^- \) that satisfies (4) for a given \( \phi \) exists? Having \( S_e \) compact and \( \beta(l_e, \phi) \) bounded does not ensure the existence of a global point of minimum in \( S_e \) [11, p. 24]. The next propositions provide sufficient conditions for the existence of \( l_e^- \) when \( \phi \) belongs to either one of two classes of systems: \( D_e' \) and \( D_d' \). To define these classes, let

\[
\phi_{t,\gamma}(T(z_1, \ldots, z_K)) := \begin{cases} 1 & \text{if } T(z_1, \ldots, z_K) > t \\ \gamma & \text{if } T(z_1, \ldots, z_K) = t \\ 0 & \text{if } T(z_1, \ldots, z_K) < t \end{cases}
\]

for some function \( T \), threshold \( t \), and parameter \( \gamma \in [0, 1] \).

**Definition of \( D_e' \):** A centralized detection system \( \phi \in D_e' \) if it decides for \( H_1 \) with probability given by the fusion function

\[
\phi_0(z_1, \ldots, z_K) = \phi_{t,\gamma}(K \prod_{i=1}^M T_{0,i}(z_{i,j}))
\]

where \( \{T_{0,i}\}_{i=1}^K \) are positive, strictly increasing, continuous functions; \( t \) is any threshold; and \( \gamma \in [0, 1] \).

**Definition of \( D_d' \):** A distributed detection system \( \phi \in D_d' \) if it decides for \( H_1 \) with probability given by the fusion function

\[
\phi_0(u_1, \ldots, u_K) = \phi_{t,\gamma}(K T_{0,i}(u_i))
\]

where \( \phi_{t,\gamma} \) is given by (7); \( \{T_{0,i}\}_{i=1}^K \) and \( t \) are as defined as in \( D_e' \); and its sensor functions are given by

\[
u_i := \nu_i(\delta_i) = \sum_{u=0}^{U_{\text{max}}} u \cdot \{t_{s,u} \leq \prod_{j=1}^M T_i(z_{i,j}) < t_{s,u+1}\},
\]

where \( \delta_i \) equals 1 if the statement is true and 0 otherwise; \( 0 = t_{s,0} < \cdots < t_{s,U_{\text{max}}} = t_{s,U_{\text{max}}+1} = \infty \); and \( \{T_{i}\}_{i=1}^K \) are positive, strictly increasing, and continuous functions.

To understand the motivation behind the definitions of \( D_e' \) and \( D_d' \), note that \( T_{0,i} \) and \( T_i \) are strictly increasing; and higher measurements favor \( H_1 \).

These classes include many systems of interest:

- The centralized system \( \phi_{\Sigma,\text{cds}} \) given by

\[
\phi_{\Sigma,\text{cds}}(z_1, \ldots, z_K) = \phi_{t,\gamma}(K T_{\text{cds}}(z_{i,j}))
\]

is equivalent to a system in \( D_e' \).

- For any \( l_e \in S_e \), the centralized system \( \phi_{(l_e),\text{cds}} \) given by

\[
\phi_{(l_e),\text{cds}}(z_1, \ldots, z_K) = \phi_{t,\gamma}(K \prod_{i=1}^M L_{(l_e)}(z_{i,j}))
\]

where \( L_{(l_e)}(z_{i,j}) := \int_{l_{i,j}}^{l_{i,j}} f_{Z_{i,j}}(z_{i,j}) / f_{Z_{i,j}}(z_{i,j}) \) with \( f_{Z_{i,j}}(z_{i,j}) \) and \( f_{Z_{i,j}}(l_{i,j}) \) being the probability density or mass functions of \( Z_{i,j} \) conditioned on \( H_0 \) and \( H_1 \) and \( \{l_e = l_e\} \) belongs to \( D_e' \) under either the Gaussian or the Poisson cases.

- The system \( \phi_{\Sigma,\text{dds}} \) with

\[
\phi_{\Sigma,\text{dds}}(u_1, \ldots, u_K) = \phi_{t,\gamma}(K \sum_{i=1}^M c_i u_i)
\]

for some constants \( \{c_i\}_{i=1}^K \) and

\[
u_i := \nu_i(\delta_i) = \sum_{u=0}^{U_{\text{max}}} u \cdot \{t_{s,u} < \sum_{j=1}^M z_{i,j} < t_{s,u+1}\}
\]

where \( \nu_i \) is any of the systems of interest described above.

C. Deterministic but Unknown Emitter Location

Although the focus is on the case where the emitter location \( l_e \) is the realization of a random variable \( L_e \), the procedures and results presented in this paper can also be applied when the emitter is at a deterministic but unknown location \( l_e \).

The goal is still to design a given system \( \phi \) to satisfy a performance requirement under the worst possible condition. For a deterministic \( l_e \), this means adopting a worst case location \( l_e^- \) that minimizes \( \beta(l_e, \phi) \).

To determine the existence of \( l_e^- \), note that a deterministic \( l_e^- \) is equivalent to a random \( L_e \) satisfying \( P_1 [L_e = l_e^-] = 1 \). Thus, \( \int_{l_e} F(z[|l_e^-|]) dP_{L_e}(l_e) = F(z[|l_e^-|]) \) and \( H_1 \) becomes \( \{z[|l_e^-|]) \) : \( l_e \in S_e \}. Even under such a constraint \( H_1 \), Propositions 1 and 2 can be used to determine the existence of \( l_e^- \).
D. Finding Least Favorable Distributions and Lower Bounds

Assuming that \( l^r_e \) that satisfies (4) exists, how can a designer find it? Finding \( l^r_e \) analytically is difficult because \( \beta(l_e, \phi) \) as a function of \( l_e \) is non-convex and numerical convex optimization methods cannot be used. Finding an approximation \( l^r_e \) is not enough since \( \beta(l^r_e, \phi) \geq \beta(l_e, \phi) \), and design that satisfies \( \beta(l^r_e, \phi) \geq \beta_{\min} \) under \( \mathbb{P}[l_e = l^r_e] = 1 \) may not satisfy the requirement under the unknown distribution.

Instead of finding \( l^r_e \), it is enough to find a tight lower bound for \( \beta(l_e, \phi) \). With such a bound, the designer is ensured that \( \phi \) satisfies the performance requirement under the LFD and under any other distribution for the emitter location.

IV. BRANCH AND BOUND ALGORITHM FOR FINDING LOWER BOUND FOR PROBABILITY OF DETECTION

To find a tight lower bound for \( \beta(l^r_e, \phi) \), the branch and bound algorithm proposed by [101] can be used:

1. Consider the region of interest \( S_e \) can be decomposed in a finite union of rectangles \( R(r) \) with disjoint interiors; and let \( \mathcal{R} \) be the set of such rectangles.
2. Pick \( \eta > 0 \), which determines how close the bound is to the global minimum when the algorithm ends.
3. For each rectangle \( R(r) \in \mathcal{R} \), let
   \[
   l_{e,i} := \arg\min_{l_e} \beta(l_e, \phi) \quad (14)
   \]
   and compute a lower bound \( L(r) \) and an upper bound \( U(r) \) for \( \beta(l_{e,i}, \phi) \).
4. Let \( L^- := \min \{ L(r) : r \in \{1, \ldots, |\mathcal{R}| \} \} \) and \( U^- := \min \{ U(r) : r \in \{1, \ldots, |\mathcal{R}| \} \). If \( U^- - L^- < \eta \), then the algorithm concludes.
5. Let \( R^- \) be any rectangle that satisfies \( L(r) = L^- \).
6. Split \( R^- \) in two rectangles along its longest side; include these two new rectangles in \( \mathcal{R} \); and remove \( R^- \) from \( \mathcal{R} \). Return to step 3.

When the algorithm concludes, adopt \( L^- \) as the lower bound for \( \beta(l^r_e, \phi) \).

If \( S_e \) cannot be decomposed in a finite union of disjoint rectangles, then adopt an enlarged region \( S^r_e \supseteq S_e \) that can be decomposed in disjoint rectangles and use the algorithm to find a lower bound for \( \beta(l_{e,i}^r, \phi) \) where \( \forall l_e \in S^r_e \), \( \beta(l_{e,i}^r, \phi) \geq \beta(l_{e,i}^r, \phi) \). Since \( l_{e,i}^r \in S_e \subseteq S^r_e \), \( \beta(l_{e,i}^r, \phi) \geq \beta(l_{e,i}^r, \phi) \) and a lower bound for \( \beta(l_{e,i}^r, \phi) \) is also a lower bound for \( \beta(l_{e,i}^r, \phi) \).

A. The Subsetting \( l^r_e \)

Although the algorithm is focused on finding a lower bound for \( \beta(l_e, \phi) \) and does not find \( l^r_e \), it provides information about \( l^r_e \) when the algorithm ends, \( l^r_e \in S^r_e \), where \( S^r_e \) is given by the union of rectangles that satisfy \( L(r) \leq U^- \).

If a more refined \( S^r_e \) is desired, substitute step 5 of the algorithm with the following step 5*: Among all \( R(r) \) satisfying \( L(r) \leq U^- \), let \( R^r \) be the subset of rectangles with the largest area; and let \( R^- \) be any of the rectangles with the highest \( U(r) \) inside \( R^r \). This modification will however increase the number of iterations of the algorithm.

B. Bounds for the Minimum Probability of Detection

At each iteration of the algorithm and for each rectangle \( R(r^r) \), it is necessary to find upper and lower bounds for \( \beta(l_{e,i}^r, \phi) \); i.e., for the minimum probability of detection when the emitter location is restricted to belong to \( R(r^r) \).

For the upper bound, one can consider the emitter location to be at any point within \( R(r^r) \) since, by definition, \( \forall l_e \in R(r^r), \beta(l_e, \phi) \geq \beta(l_{e,i}^r, \phi) \). Thus, let \( l_{e,i}^r \) be the point in which the diagonals of \( R(r^r) \) intersect; and let \( U(r^r) = \beta(l_{e,i}^r, \phi) \).

Finding a lower bound is not as simple. For this, the model of Section II is augmented as follows: instead of considering a single emitter, consider \( K \) emitters; and instead of considering that all sensors measure a common emitter, consider that sensor \( i \) measures only emitter \( i \). Let \( L_{e,i} := (l_{e,i,1}, \ldots, l_{e,i,K}) \) and \( L_e := (l_{e,1}, \ldots, l_{e,K}) \) be the random variables and realizations for the locations of \( K \) emitters; let \( Z_{i,j} \) be a measurement of sensor \( i \) having a distribution that depends on \( ||l_i - l_{e,j}|| \); and let \( U_{i} \) be the sensor output. Note that if \( l_{e,i,1} = \cdots = l_{e,i,K} \) almost surely, then \( Z_{i,j} \) and \( U_i \) have the same distribution as \( Z_{i,j} \) and \( U_i \) of the original model. Let \( \beta(l_e, \phi) \) be the probability of \( \phi \) deciding for \( H_1 \) under \( H_1 \) and \( \mathbb{P}[l_e = l_i] = 1 \). When \( l_e = (l_{e,1}, \ldots, l_{e,K}) \), \( \beta(l_e, \phi) = \beta(l_{e,i}, \phi) \). Proposition 3 shows how the \( K \) emitters assist in finding a lower bound.

Proposition 3: For any given \( R(r) \), let \( l_{e,i}^r \) be the location in \( R(r) \) that maximizes the distance to sensor \( i \); i.e.,
\[
 l_{e,i}^r := \arg\max_{l_e} ||l_i - l_e||, \quad (15)
\]
Assume that \( \forall z, F(z|x) \) is a nonincreasing function of \( x \). If \( \phi \in D_{l_{e,i}^r}^l \), then \( \beta(l_{e,i}^r, \phi) \geq \beta(l_{e,i}^r, \phi) \).

Proof: Consider that \( \phi \in D_{l_{e,i}^r}^l \). For any \( l_e \in S_c \), \( \beta(l_e, \phi) = P_1[\phi_{0}(U_{e,1}, \ldots, U_{e,K}) = 1 | l_e = l_i] = P_1[\phi_{0}(U_{e,1}, \ldots, U_{e,K}) = 1 | l_e = (l_{e,1}, \ldots, l_{e,l_i})]; \) and to reach the conclusion, it is proven that, \( \forall l_{e,1}, \ldots, l_{e,K} \in R(r) \) and \( \forall i \),
\[
 P_1[\phi_{0}(U_{e,1}, \ldots, U_{e,K}) = 1 | l_e = (l_{e,1}, \ldots, l_{e,K})] \geq P_1[\phi_{0}(U_{e,1}, \ldots, U_{e,K}) = 1 | l_e = (l_{e,1}, \ldots, l_{e,K})]; \quad (16)
\]
i.e., the conditional probability of detection does not increase if \( l_{e,i} \) is replaced by \( l_{e,i}^r \); and the conclusion follows from (16) because each \( l_{e,i} \) can be replaced by \( l_{e,i}^r \) one by one until all \( l_{e,i} \) are replaced. Relation (16) is proven for \( i = 1 \); and the same proof can be used for any other \( i \).

To prove (16), let \( U_{2K} = (U_{2,1}, \ldots, U_{2,K}) \), let \( u_{2K} = (u_2, \ldots, u_K) \), and observe that
\[
P_1[\phi_{0}(U_{1,1}, \ldots, U_{1,K}) = 1 | l_e = (l_{e,1}, \ldots, l_{e,K})] = \sum_{u_{2K}} P_1[\phi_{0}(U_{1,1}, U_{2K}) = 1 | l_{e,1} = l_e, U_{2K} = u_{2K}]
\]
\[
P_1[\phi_{0}(U_{1,1}, U_{2K}) = 1 | l_{e,1} = l_e, U_{2K} = u_{2K}]; \quad (17)
\]
1. \( \phi \) does not attempt to detect each one of the \( K \) emitters, and still decides between \( H_0 \) (no emitter present) and \( H_1 \) (a single emitter somewhere in \( S_c \)).

Conditioned on \( \{l_e = l_i\} \), \( H_{1} \) corresponds to \( Z_{i,j} \) distributed according to \( F(z|l_i - l_{e,j}) \). Considering \( K \) sensors individually sensing \( K \) emitters at different positions has the only effect of changing the distributions of \( \{Z_{i,j}\}_{j=1}^{K} \) in order to reach a lower bound for \( \beta(l_{e,i}^r, \phi) \).
continuous function of and continuous, it also follows when \( T \) is strictly increasing and continuous, (19) is equivalent to
\[
\phi_0(U'_1, u_{2K}) = 1 \iff T_{0,i}(U'_1) = \prod_{i=2}^{K} T_{0,i}(u_i) \in I_y,
\]
where \( y \) is the realization of a Bernoulli random variable with \( P[Y = 1] = \gamma, I_0 = (t, \infty), \) and \( I_1 = [t, \infty) \).

Consider \( y = 1 \). Since \( T_{0,i} > 0 \), (19) is equivalent to
\[
T_{0,i}(U'_1) \geq t \prod_{i=2}^{K} T_{0,i}(u_i)
\]
and since \( T_{0,i} \) is strictly increasing and continuous, (19) is equivalent to \( U'_i \geq u \) for some \( u \) that depends on \( t \) and \( u_{2K} \). Thus, (18) follows from
\[
\forall u, P_1[|U'_1| \geq u|L_{e,1} = l_e] \geq P_1[|U'_i| \geq u|L_{e,1} = l_{e,1}^{(r)}]
\]
and once (20) is proven, it follows that
\[
\forall u, P_1[|U'_1| > u|L_{e,1} = l_e] \geq P_1[|U'_i| > u|L_{e,1} = l_{e,1}^{(r)}]
\]
(12, Lemma F.3) and (18) also follows for the case \( y = 0 \).

To prove (20), recall from (10) that \( U'_i = \phi_i(Z'_i) \), a right-continuous function of \( \prod_{j=1}^{M} T_i(Z'_{1,i}) \), and (20) follows if
\[
\forall t, P_1 \left[ \prod_{j=1}^{M} T_1(Z'_{1,j}) \geq t|L_{e,1} = l_e \right] \geq P_1 \left[ \prod_{j=1}^{M} T_1(Z'_{1,j}) \geq t|L_{e,1} = l_{e,1}^{(r)} \right].
\]

To prove (21), recall that \( \{Z'_{1,j}\}_{i=1}^{M} \) are conditionally i.i.d.; thus, using Lemma 2 of [13], (21) follows from
\[
\forall t, P_1[|T_1(Z'_{1,j})| > t|L_{e,1} = l_e] \geq P_1[|T_1(Z'_{1,j})| > t|L_{e,1} = l_{e,1}^{(r)}].
\]

To prove (22), recall that \( T_1 \) is increasing and continuous; thus, \( T_1(Z'_{1,j}) \geq t \leftrightarrow Z'_{1,j} \geq z_i \) for some \( z_i \); and (22) follows from
\[
\forall z, P_1[|Z'_{1,j}| \geq z|L_{e,1} = l_e] \geq P_1[|Z'_{1,j}| \geq z|L_{e,1} = l_{e,1}^{(r)}].
\]

Finally, (23) follows because
\[
\forall z, P_1[|Z'_{1,j}| \geq z|L_{e,1} = l_e] = F(z|\xi(||l_1 - l_e||)) \geq F(z|\xi(||l_1 - l_{e,1}^{(r)}||)) = P_1[|Z'_{1,j}| \geq z|L_{e,1} = l_{e,1}^{(r)}];
\]
where (24) follows from the definition of \( l_{e,1}^{(r)} \), the assumption that \( F(z|x) \) is a nonincreasing function of \( x \) for any \( z \), and the assumption that \( \xi \) is nonincreasing.

When \( \phi \in \mathcal{D}_p \), replace \( U_i, U'_i, U_{2K} \), and \( u_{2K} \) by \( Z_i, Z'_i, Z'_{2K} = (Z_2, \ldots, Z_{2K}) \), and \( z_{2K} = (z_2, \ldots, z_{2K}) \) in all equations up to (20); replace the summation in (17) by an integral over the joint distribution of \( Z'_{2K} \) conditioned on \( \{L_{e,1} = l_{e,1}^{(r)}\}_{i=1}^{K} \); replace \( T_{0,i}(U'_1) \) and \( T_{0,i}(u_i) \) by \( \prod_{j=1}^{M} T_{0,i}(Z_{1,j}) \) and \( \prod_{j=1}^{M} T_{0,i}(Z'_{1,j}) \) in (19); and the conclusion follows if, \( \forall l_e \in R^{(r)} \) and \( \forall t, P_1 \left[ \prod_{j=1}^{M} T_{0,i}(Z'_{1,j}) \geq t|L_{e,1} = l_e \right] \geq P_1 \left[ \prod_{j=1}^{M} T_{0,i}(Z_{1,j}) \geq t|L_{e,1} = l_{e,1}^{(r)} \right] \), which is (21) with \( T_1 \) replaced by \( T_0 \). Since (21) was proven for any \( T_1 \) increasing and continuous, it also follows when \( T_1 \) is replaced by \( T_0 \), which is also increasing and continuous.

Intuitively, Proposition 3 means that a lower bound for \( \beta(l_{e,1}^{(r)}, \phi) \) can be found by considering the greatest distance between a point in \( R^{(r)} \) and each sensor when considering the distribution of the sensor’s measurements. It is possible to show that \( F(z|x) \) is a nonincreasing function of \( x \) for either the Gaussian or the Poisson cases of Section II.

C. Termination Condition

The next proposition shows that, as the rectangles’ sides converge to zero, the difference between the chosen upper and lower bounds converges to zero uniformly, which is a sufficient condition for the termination of the algorithm [10].

**Proposition 4:** Consider that \( \phi \in \mathcal{D}_p \cup \mathcal{D}_d \) and assume the conditions of Propositions 1 or 2 accordingly. Under the conditions of Proposition 3, let \( \text{size}(R^{(r)}) \) denote the largest side of \( R^{(r)} \). For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( R^{(r)} \) with \( \text{size}(R^{(r)}) < \delta \), \( \beta(l_{e,1}^{(r)}, \phi) - \beta(l_{e,1}^{(r)}, \phi) < \epsilon \).

**Proof:** From \( \beta(l_{e,1}^{(r)}, \phi) = \beta(l_{e,1}^{(r)}, \ldots, l_{e,1}^{(r)}, \phi) \), it is enough to prove that \( \beta(l_{e,1}^{(r)}, \phi) \) is a continuous function of \( l_e \) because continuous functions on a compact \( S_e \subset R^2 \) are uniformly continuous [14, p. 198]. To prove that \( \beta(l_{e}, \phi) \) is continuous on \( l_e \), pick any sequence \( l_{e}^{(n)} \to l_{e}^{*} \) in \( S_e \), and follow similar steps as in the proof of Proposition 1 and 2. \( \square \)

D. Number of Measurements Required

Using the branch and bound procedure, the number of measurements \( M \) required to satisfy the minimum probability of detection \( \beta_{\min} \) for a maximum probability of false alarm \( \alpha_{\max} \) is determined as follows: for a given \( \phi \) with sensors at \( \{l_i\}_{i=1}^{K} \), build an increasing sequence of values for \( M \). For each \( M \), apply the branch and bound algorithm to obtain the lower bound for \( \beta(l_{e,1}^{(r)}, \phi) \); and adopt the smallest \( M \) that produces a lower bound greater than \( \beta_{\min} \).

V. EXAMPLES FOR DETECTING A SINGLE EMITTER

Let \( K = 4 \) sensors at \( l_1 = (0, 0), l_2 = (0, 2), l_3 = (2, 2), \) and \( l_4 = (2, 0) \) define the vertices of a square region \( S_e \).

Let \( F(z|x) \) be the Poisson c.d.f. with parameter \( \lambda + x \); i.e., from (1), \( Z_{i,j} \) is Poisson distributed with parameter \( \lambda \) under \( H_0 \) and with parameter \( \lambda + \xi(||l_i - l_e||) \) under \( H_1 \), when conditioned on \( L_{e} = l_e \). Assume \( \lambda = 8 \) and \( \xi(d) = 1/d^2 \) if \( d \geq 0.001 \) and \( \xi(d) = 10^6 \) if \( d < 0.001 \). Considering the dimensions in kilometers, these values are similar to the values measured in [11] in a system to detect point radiation sources.

Consider the centralized system \( \phi_0^{\Sigma, eds} \) defined by (11). For each \( M, t \) and \( \gamma \) were adjusted to satisfy \( \alpha(\phi_0^{\Sigma, eds}) = 0.05 \); and \( \alpha(\phi_0^{\Sigma, eds}) \) and \( \beta(\phi_0^{\Sigma, eds}) \) were computed using the distribution of \( \sum_{i=1}^{K} \sum_{j=1}^{M} Z_{i,j} \), which is Poisson with parameter \( K \cdot M \cdot \lambda \) under \( H_0 \) and parameter \( \sum_{i=1}^{K} M \cdot (\lambda + \xi(||l_i - l_e||)) \) under \( H_1 \) when conditioned on \( L_{e} = l_e \).

Fig. 2a shows the lower bounds for \( \beta(l_{e,1}^{(r)}, \phi_0^{\Sigma, eds}) \) obtained by the algorithm for various \( M \). The precision was set to \( \eta = 0.001 \); i.e., \( \beta(l_{e,1}^{(r)}, \phi_0^{\Sigma, eds}) \) at each \( M \) is greater than the lower bounds obtained by at most 0.001. Assuming \( \beta_{\min} = 0.95 \), \( M \geq 90 \) is required to satisfy \( \beta(l_{e,1}^{(r)}, \phi_0^{\Sigma, eds}) \geq 0.95 \).
What if the unknown emitter location distribution is assumed to be the uniform distribution \((P^u_{\nu})\)? Fig. 2a shows that \(M \approx 55\) is enough to satisfy \(\beta(P_{L_e}^e, \phi^{(maj)}) \geq 0.95\); however, the uniform distribution may not be the distribution for the emitter location and the performance may fail to meet the requirement; i.e., the resulting probability of detection under the actual distribution may be as low as 0.85 when \(M = 55\).

Consider now the distributed system \(\phi^{(maj)}\), where sensor \(i\) sends \(U_i = 1\) if \(\sum_{j=1}^{M} Z_{i,j} > t_{s,1}\) or \(U_i = 0\) otherwise; and the fusion function uses a randomized majority rule. More precisely, \(\phi^{(maj)}\) is given by \(\phi^{S,cds}\) defined in Section III-B with \(U_{\text{max}} = 1\), \(c_i = 1\), and \(t = 2\). For each \(M\), \(t_{s,1}\) and \(\gamma\) are adjusted so that \(\alpha(\phi^{(maj)}) = 0.05\).

Fig. 2b shows the lower bounds for \(\beta(l_e^-, \phi^{(maj)})\) for various \(M\); and \(M \geq 600\) is required to satisfy \(\beta(l_e^-, \phi^{(maj)}) \geq 0.95\). If the uniform distribution is adopted instead, \(M \approx 300\) is enough to satisfy \(\beta(P_{L_e}^e, \phi^{(maj)}) \geq 0.95\); however, the resulting probability of detection under the actual distribution may be as low as 0.81.

Consider now the distributed system \(\phi^{(or)}\) that uses the same sensor functions as \(\phi^{(maj)}\) but the fusion center decides for \(H_1\) only if \(\sum_{i=1}^{4} U_i \geq 0\). For each \(M\), \(t_{s,1}\) is adjusted so that \(\alpha(\phi^{(or)}) \leq 0.05\). The algorithm found lower bounds for \(\beta(l_e^-, \phi^{(or)})\) for various \(M\); and \(M \geq 180\) is required to satisfy \(\beta(l_e^-, \phi^{(or)}) \geq 0.95\). If the uniform distribution is adopted instead, \(M \approx 80\) is enough to satisfy \(\beta(P_{L_e}^e, \phi^{(or)}) \geq 0.95\); however, the resulting probability of detection under the actual distribution may be as low as 0.68.

Fig. 3 shows the rectangles treated by the algorithm and the resulting subsets \(S_e\) that contain \(l_e^-\) for each of the systems considered. Fig. 3a shows that, for \(\phi^{(or)}\), \(l_e^-\) is close to the center of \(S_e\). This is expected because such a \(l_e^-\) minimizes the probability that any sensor reports \(U_i = 1\).

Fig. 3b shows that any distribution that Places \(L_e\) in each of the 4 vertices with probabilities that add to 1 is an LFD for \(\phi^{(maj)}\). Although \(L_e\) at any of the vertices maximizes the probability that the corresponding sensor reports \(U_i = 1\), it also minimizes the probability that the remaining sensors report \(U_i = 1\) and minimizes the probability of detection since \(\phi^{(maj)}\) only decides for \(H_1\) if at least two sensors report \(U_i = 1\).

However, as shown in Fig. 3c, the resulting \(S_e^-\) for \(\phi^{S,cds}\) does not allow a designer to determine whether \(l_e^-\) is close to the center or to the boundary of the region. This example highlights the point that the algorithm does not focus on finding \(l_e^-\). Using the modification proposed in Section IV-A, one obtains the more refined \(S_e^-\) shown in Fig. 3d, which shows that \(l_e^-\) is in fact close to the center of \(S_e\).

VI. DETECTION OF MULTIPLE EMITTERS

This section shows that the approach of adopting an LFD for the emitter location can also be applied when the region has multiple emitters; and that considering a single emitter is the worst case for the detection system.

Assume that \(S_e\) under \(H_1\) has a random number \(E \geq 1\) of emitters, each at a random location \(L_e\) with an unknown distribution \(P_{L_e}^{(E)}\). Consider that \(\{L_e\}_{e=1}^{E}\) are independent. Let \(\beta^{(E)}(P_{L_e}^{(1)}, \ldots, P_{L_e}^{(E)}, \xi, \phi)\) represent the probability of \(\phi\) deciding for \(H_1\) when at least one emitter is present and \(\{L_e\}_{e=1}^{E}\) are distributed according to \(\{P_{L_e}^{(E)}\}_{e=1}^{E}\) and the decay function is \(\xi\). Note that \(\beta^{(1)}(P_{L_e}^{(1)}, \xi, \phi) := \beta(P_{L_e}^{(1)}, \phi)\) defined in Section II, the only difference being that the decay function \(\xi\) is made explicit here. As in Section II, the probability of deciding for \(H_1\) conditioned on \(H_1\) and \(\{L_e\}_{e=1}^{E} = l_e^e\) for some \(l_e^e\) is represented by \(\beta^{(E)}(P_{L_e}^{(1)}, \ldots, l_e^e, \ldots, P_{L_e}^{(E)}, \xi, \phi)\).

A set of distributions \(P_{L_e}^{(1)}, \ldots, P_{L_e}^{(E)}\) is defined to be least favorable for \(\phi\) if \(\forall \phi \in P_{L_e}^{(1)}, \ldots, P_{L_e}^{(E)}\), \(\beta^{(E)}(P_{L_e}^{(1)}, \ldots, l_e^e, \ldots, P_{L_e}^{(E)}, \xi, \phi) \geq \beta^{(E)}(P_{L_e}^{(1)}, \ldots, P_{L_e}^{(E)} - , \xi, \phi)\).

\(\xi\) does not attempt to detect all of the \(E\) emitters and still decides between \(H_0\) and \(H_1\) (at least one emitter present in \(S_e\)). Under \(H_1\), the distribution of each measurement \(Z_{i,j}\) is influenced by all of the \(E\) emitters.
**Proposition 5:** Let $S_e$ contain $E$ emitters under $H_1$, where $E$ is a random variable with unknown distribution $P_E$. If $F(z|x)$ is nonincreasing with $x$; if

$$
P_1[z_{i,j} \leq z_{i,j}|\{l_e^{(e)}=l_e^{(e)}\}_{e=1}^E] = F(z_{i,j}) \sum_{e=1}^E \xi(||l_e-l_e^{(e)}||),$$

and if there exists $l_e^-$ that satisfies (4) for a system $\phi$ for any bounded, positive, nonincreasing, and continuous decay function $\xi$ when $E = 1$; then

1) the set of distributions $\{P_{l_e}^{(e)}\}_{e=1}^E$ such that $P_1[l_e^{(e)} = l_e^-] = 1$ for any $e$ is least favorable for $\phi$; and

2) the probability of detection of $\phi$ under the set of LFDs is lowest when $P_1[E = 1] = 1$.

**Proof:** Please refer to the appendix.

It is possible to show that the conditions of Proposition 5 are satisfied for either the Gaussian or the Poisson cases when the effect of the emitters is additive. Such additive effect is present when detecting point radiation sources: measurements are Poisson with parameter $\lambda$ when conditioned on $H_0$ and with parameter $\lambda + \sum_{e=1}^E \xi(||l_e-l_e^{(e)}||)$ when conditioned on $H_1$ and on $E$ radiation sources at $l_e^{(1)}, \ldots, l_e^{(E)}$.

**VII. GENERALIZED LIKELIHOOD RATIO TESTS**

Consider that the system $\phi$ has an estimation subsystem that, at each decision interval, generates an estimate $\hat{l}_e$ for the emitter location; and consider that $\hat{l}_e$ is used by the detection subsystem to decide between $H_0$ and $H_1$. This section shows that the approach presented in previous sections can also be used to reach conservative designs for such systems.

For each $\hat{l}_e$, let $\phi^\hat{l}_e$ be the detection subsystem that uses a fusion function based on the Generalized Likelihood Ratio Test (GLRT) [7, p. 200]. In a centralized system, $\phi^\hat{l}_e = \phi^{(L_e),cds}$ given by (12). In a distributed system with given sensor functions, the GLRT fusion function depends on the ratio between the distribution of $\{U_{l_e}\}_{l_e=1}^K$ under $H_1$ and $\{\hat{l}_e\}$ and the distribution of $\{U_{l_e}\}_{l_e=1}^K$ under $H_0$. When $U_{max} = 1$, such a fusion function is equivalent to (13) for certain coefficients $\{c_{i,j}\}_{i=1}^K$ [6, p. 185]; and both $\phi^{(L_e),cds}$ and $\phi^{(L_e),dds}$ of Section III-B can be examples of $\phi^\hat{l}_e$.

**A. Constrained Least Favorable Distributions**

If $\hat{l}_e$ were exact, then the conditional distribution of $Z_{i,j}$ under $H_1$ would be $F(z|\xi(||l_e-\hat{l}_e||))$; and the system could be designed without the methods presented here; however, estimation subsystems cannot remove all the uncertainty about the emitter location; $\hat{l}_e$ may be different from the actual emitter location; and $M$ computed under $F(z|\xi(||l_e-\hat{l}_e||))$ may not be high enough to satisfy the detection requirement.

Uncertainties about parameter estimates are commonly modeled using confidence intervals [15, p. 202]; however, since $\hat{l}_e$ is a location in the plane, this paper models the uncertainty about $\hat{l}_e$ using a q-confidence rectangle $R_{e}$. More precisely, the q-confidence rectangle $R_{e}$ is a compact subset of $S_e$ that contains the actual $l_e$ with probability at least $q \epsilon (0,1)$; i.e., $P_{l_e}[R_e] \geq q$. In addition to $\hat{l}_e$, the estimation subsystem also supplies the q-confidence rectangle $R_{e}$ to the detection subsystem. Note that the distribution of the emitter location is still considered unknown; however, the system gained the information that $P_{l_e}$ satisfies $P_1[L_e \in R_e] \geq q$.

To ensure that the detection requirement is satisfied under this uncertainty, it is proposed here that the detection subsystem determine a conservative $M$ by using a constrained LFD: for a $\phi$ satisfying $\alpha(\phi) \leq \alpha_{max}$ and a q-confidence rectangle $R_{e}$, $P_{l_e}^{(\phi)}$ is a constrained LFD for $\phi$ if $P_{l_e}^{(\phi)}[R_e] \geq q$ and

$$\forall P_{l_e}^{(\phi)} \text{ such that } P_{l_e}^{(\phi)}[R_e] \geq q, \beta(P_{l_e}^{(\phi)}; \phi) \geq \beta(P_{l_e}^{(\phi)}; \phi).$$

As before, adopting $P_{l_e}^{(\phi)}$ ensures a conservative design; i.e., if the detection subsystem uses $\phi^\hat{l}_e$ with $M$ measurements such that $\beta(P_{l_e}^{(\phi)}; \phi) \geq \beta_{min}$, then $\beta(P_{l_e}^{(\phi)}; \phi) \geq \beta_{min}$ for any $P_{l_e}$ that satisfies $P_1[L_e \in R_e] \geq q$.

**B. Finding Constrained Least Favorable Distributions**

For a given q-confidence rectangle $R_{e}$, an estimate $\hat{l}_e$, and a detection subsystem $\phi^\hat{l}_e$, if there exists $l_e^- \in R_e$ that satisfies

$$\forall l_e \in R_e, \beta(l_e, \phi^\hat{l}_e) \geq \beta(l_e^-, \phi^\hat{l}_e),$$

and a point $l_e^- \in S_e$ that satisfies

$$\forall l_e \in S_e, \beta(l_e, \phi^\hat{l}_e) \geq \beta(l_e^-, \phi^\hat{l}_e),$$

the system is conservative.
then for any $P'_{L_e}$ such that $P'_{L_e}[R_e] \geq q$, $\beta(P'_{L_e}, \varphi^c)$ equals
\[
\int_{R_e} \beta(t_e, \varphi^c) dP'_{L_e}(t_e) + \int_{S_e-R_e} \beta(t_e, \varphi^c) dP'_{L_e}(t_e)
\geq \int_{R_e} \beta(t_e^{-1}, \varphi^c) dP'_{L_e}(t_e) + \int_{S_e-R_e} \beta(t_e, \varphi^c) dP'_{L_e}(t_e)
\geq \beta(t_e^{-1}, \varphi^c) \cdot q + \beta(t_e, \varphi^c) \cdot (1 - q);
\] (28)
i.e., the distribution that places the emitter at $t_e^{-1}$ with probability $q$ and at $t_e$ with probability $1 - q$ satisfies (25) and is a constrained LFD for $\varphi^c$.

Since both $R_e$ and $S_e$ are compact sets, the existence of $t_e^{-1}$ and $t_e$ follows from Propositions 1 and 2; and tight lower bounds for $\beta(t_e^{-1}, \varphi^c)$ and $\beta(t_e, \varphi^c)$ can be found using the algorithm of Section IV. Since $t_e$ is defined as (27), the lower bound for $\beta(t_e^{-1}, \varphi^c)$ can be obtained using the algorithm without any change. To find the lower bound for $\beta(t_e^{-1}, \varphi^c)$, the same algorithm is applied with the only change that, in step (1), $R = \{R_e\}$.

C. Example of Detection with GLRT System

Consider the same $\xi$ and $S_e$ as in the example of Section V with $K = 4$ sensors at each vertex. Assume the Gaussian model where $Z_{i,j}$ is Gaussian distributed with variance $\sigma^2 = 8$ and mean 0 under $H_0$ and mean $\xi(\|l_i-l_j\|)$ under $H_1$ when conditioned on $\{L_e=l_e\}$.

Consider that the estimation subsystem provides $\hat{t}_e = (0.7, 0.5)$ to the detection subsystem, which uses the centralized GLRT system $\phi(\hat{t}_e), cdS$ defined by (12). Under the Gaussian model, $\phi(\hat{t}_e), cdS$ decides for $H_1$ if $\sum_{i=1}^{K} \sum_{j=1}^{M} c_i Z_{i,j} > t$ for $c_i = \xi(\|l_i-l_j\|)$ and $t = \sqrt{M\sigma^2 \sum_{i=1}^{K} c_i^2 \cdot Q^{-1}(\alpha_{\max})}$, where $Q$ is the complementary c.d.f. of the Gaussian distribution with mean 0 and variance 1.

Assume that the estimation subsystem also provides the 0.95-confidence rectangle $R_e$ defined by $[0.7-r, 0.7+r] \times [0.5-r, 0.5+r]$, where different values of $r$ will be considered.

For the given $\hat{t}_e$ and $R_e$, the lower bound for the probability of detection for various $M$ under $\alpha(\phi(\hat{t}_e), cdS)=0.05$ is obtained by first using the proposed algorithm with $R = \{S_e\}$ to obtain a lower bound for $\beta(\hat{t}_e, \varphi(\hat{t}_e), cdS)$; and then using the algorithm with $R = \{0.7-r, 0.7+r\} \times [0.5-r, 0.5+r]$ to obtain a lower bound for $\beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e), cdS)$. The precision was set to $\eta=0.001$. When searching for the lower bound, the algorithm uses $\beta(\hat{t}_e, \varphi(\hat{t}_e), cdS) = Q((t-M \sum_{i=1}^{K} c_i \xi(\|l_i-l_j\|)) / \sqrt{M\sigma^2 \sum_{i=1}^{K} c_i^2})$.

When computing the lower bound for $\beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e), cdS)$, the algorithm only considers rectangles in $R_e$; and, as shown in Fig. 4, the resulting $S_e^{-}$ shows that $\hat{t}_e^{-1}$ is close to $(1.0, 0.8)$ when $r=0.3$. This is expected since $(1.0, 0.8)$ is the point in $R_e$ that maximizes the distance to the sensor with highest $c_i$.

Fig. 5 shows the lower bounds for $\beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e), cdS)$, $0.95 + \beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e), cdS) \cdot 0.05$ and the performance if $\hat{t}_e$ is exact ($\beta(\hat{t}_e, \varphi(\hat{t}_e), cdS)$) as a function of $M$ for various $r$. As expected, the performance improves as $r \rightarrow 0$ since $\hat{t}_e^{-1}$ gets close to $\hat{t}_e$ for which the system is designed.

Fig. 5 also shows that, if a GLRT system is built using $\hat{t}_e$ and the emitter is assumed to be at $\hat{t}_e$ during the performance evaluation; i.e., the estimate $\hat{t}_e = (0.7, 0.5)$ is assumed to be exact; then one could decide that $M=40$ is enough to satisfy the minimum detection requirement of $\beta_{\min}=0.95$. However, if the estimation procedure has errors, then the resulting design may fail to satisfy the requirement. For instance, if the estimation uncertainty is such that $r=0.3$, then the resulting probability of detection with $M=40$ may be as low as 0.64. In this case, to ensure that the probability of detection is greater than $\beta_{\min}=0.95$, $M \geq 110$ should be used instead.

D. Number of Measurements Required by GLRT System

Since a different decision statistic is used for each $\hat{t}_e$, the number of measurements $M$ required to satisfy the detection requirement may change with the estimate $\hat{t}_e$.

If enough processing power is available, the required $M$ can be determined immediately after the estimation subsystem supplies $\hat{t}_e$ and the q-confidence interval $R_e$ and before the end of the decision interval. Such a determination can be done as exemplified in Section VII-C; i.e., lower bounds for $\beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e))$ and $\beta(\hat{t}_e^{-1}, \varphi(\hat{t}_e))$ are computed for each $M$ and
the smallest $M$ that satisfies $\beta(l_{e}^{-1}, \phi(l_{e}^{s})) \cdot q + \beta(l_{e}^{s}, \phi(l_{e}^{s})) \cdot (1 - q) \geq \beta_{\text{min}}$ is adopted.

If enough processing power is not available, the designer can compute ahead of time the required $M$ to satisfy the detection requirement for a finite set of possible estimates and q-confidence rectangles. Although there are uncountable possible estimates and q-confidence rectangles, the designer can configure the estimation subsystem to only supply quantized estimates $l_{e}^{#}$. For instance, the estimation subsystem can be configured to only provide estimates $l_{e}^{#}$ that belong to a fine granular grid, and the number of possible estimates becomes finite. Thus, the designer can compute ahead of time the required $M$ to satisfy the detection requirement for each possible $l_{e}^{#}$, and configure the detection subsystem with a look-up table that maps $l_{e}^{#}$ into the required $M$. Likewise, since the required $M$ also depends on the q-confidence rectangle, the estimation subsystem can be configured to only provide one out of a finite set $\mathcal{R}_{e}^{#}$ of possible q-confidence rectangles around $l_{e}^{#}$. Thus, instead of supplying any q-confidence rectangle $R_{e}$, the estimation subsystem supplies the smallest q-confidence rectangle $\mathcal{R}_{e}^{#}$ that satisfies $R_{e} \subset \mathcal{R}_{e}^{#} \in \mathcal{R}_{e}^{#}$. The detection subsystem is then equipped with a finite number of look-up tables mapping $l_{e}^{#}$ into the required $M$ for each possible q-confidence rectangle $\mathcal{R}_{e}^{#}$. Note that if $\beta(l_{e}^{-1}, \phi(l_{e}^{s})) \cdot q + \beta(l_{e}^{s}, \phi(l_{e}^{s})) \cdot (1 - q) \geq \beta_{\text{min}}$ for the computed $M$, where $l_{e}^{-1} \in \mathcal{R}_{e}^{#}$, then the GLRT system $\phi(l_{e}^{s})$ also satisfies the detection performance for any emitter location distribution $P_{Le}$, that satisfies $P[L_{e} \in R_{e}] \geq q$. If the deviation of $M$ values among all the look-up tables is small, the designer can simply configure the detector to always use the largest $M$ among all tables.

### VIII. RELATED APPROACHES

The use of LFDs in the design of sensor detection systems was also considered by [13], [16], [17], which determined general conditions in which LFDs for centralized systems are also least favorable for distributed systems. Their results require conditionally i.i.d. measurements and do not consider the influence of the emitter location in the measurements.

The use of LFDs for emitter locations was studied by the authors in [8] when sensor locations are random. Sensors at deterministic and known locations, as in this paper, were considered in a preliminary form by this author in [9], where it was shown that finding an LFD for the emitter location of certain systems is equivalent to solving the Obnoxious Facility Location Problem from the field of operations research. Unfortunately, the equivalence does not hold for important systems of interest, such as $\phi^{(m_{o})}$ described in Section V.

There is a similarity between the approach proposed here and the approach proposed in [18], in which the authors use the minimum probability of detection as the emitter location varies in the region as a performance metric. The authors considered only the case of a deterministic unknown emitter location, focused on strategies for sensor placement, and did not address methods to obtain a lower bound for the probability of detection under the worst possible location.

The worst case for a detection system was also considered in [19], where a transmitter at a known location is detected by sensors at worst possible locations.

### IX. CONCLUSIONS AND DISCUSSION

The main conclusion of this paper is that, when designing a given sensor detection system where measurements are conditionally dependent due to randomness in the emitter location and the hypothesis test is composite due to the lack of knowledge about the emitter location’s distribution, a designer can adopt the least favorable distribution for the emitter location to avoid the difficulties associated with the conditional dependence and the composite hypothesis. In several scenarios of interest, the least favorable distribution places the emitter at a particular location with probability 1, making the measurements conditionally independent and the hypothesis test simple.

In addition to facilitating the design, the adoption of the least favorable distribution for the emitter location produces a conservative design; i.e., if a system requires a certain number of measurements to satisfy a prescribed detection performance while considering the least favorable distribution, then such a number is enough to satisfy the prescribed performance for any other distribution. Assuming any other distribution for the emitter location, such as the uniform distribution, is not recommended because the resulting design may not satisfy the prescribed performance under the actual distribution; i.e., the resulting design may underestimate the number of measurements required to satisfy the prescribed performance.

The design under the least favorable distribution can be found by using a known branch and bound algorithm with the upper and lower bounds identified in this paper.

Even in systems that use the GLRT detector, in which the emitter location is first estimated to build the detector, the approach presented here can be used to model the remaining uncertainty in the emitter location and produce conservative designs for the GLRT detector. It is however necessary to determine the confidence region for the estimate, which is not addressed here and is an avenue for future research. Another area of research is to determine whether least favorable distributions can be used to generate emitter location estimates with a guaranteed confidence region.

Other avenues for future research include: extending results to sensor systems with censoring and determining the maximin detector, which is the detection system that maximizes the detection performance under the least favorable distribution.
where \( \Rightarrow \) indicates convergence in distribution. Since \( T_{0,i} \) is positive and continuous, \( \log T_{0,i} \) is continuous, and \( \log T_{0,i}(Z_{i,j}^{(n)}) \Rightarrow \log T_{0,i}(Z_{i,j}) \). Since \( \{Z_{i,j}^{(n)}\}_{j=1}^{K} \) are conditionally independent given \( \{L_{e} = l_{e}^{(n)}\} \),
\[
\sum_{i=1}^{K} \sum_{l_{e}^{(n)}} \log T_{0,i}(Z_{i,j}^{(n)}) = \sum_{i=1}^{K} \sum_{l_{e}^{(n)}} \log T_{0,i}(Z_{i,j}^{(n)})
\]
and considering the exponential of this sequence,
\[
\prod_{i=1}^{K} T_{0,i}(Z_{i,j}^{(n)}) = \prod_{i=1}^{K} T_{0,i}(Z_{i,j})
\]
If the distribution of \( Z_{i,j} \) has no point masses for any \( l_{e} \), \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}) \) also has no point masses since \( T_{0,i} \) is continuous and strictly increasing. Thus, the c.d.f. of \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}^{(n)}) \) is continuous and \( g(t^{(n)}) \rightarrow g(t^{*}) \) for each \( I = (t, \infty) \) or \( I = [t, \infty) \).

If the distribution \( Z_{i,j} \) is a p.m.f, the distribution of \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}) \) is also a p.m.f. for any \( l_{e} \); and, by assumption, \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}^{(n)}) \in B \) and \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}^{(n)}) \in B \), where \( B \) has no accumulation points. Using Lemma 4 of [9], it follows that the c.d.f. of \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}^{(n)}) \) converges pointwise to \( \prod_{i=1}^{K} T_{0,i}(Z_{i,j}) \) and it follows that \( g(t^{(n)}) \rightarrow g(t^{*}) \) for each \( I = (t, \infty) \) or \( I = [t, \infty) \).

**Proof of Proposition 5**

Since the probability of detection is
\[
\sum_{n=1}^{\infty} \beta^{(n)}(P_{L_{1}^{(n)}}, \ldots, P_{L_{N}^{(n)}}, \xi, \phi) \cdot P_{1}[E = e],
\]
the first conclusion follows if \( \forall e, \beta^{(e)}(P_{L_{1}^{(n)}}, \ldots, P_{L_{N}^{(n)}}, \xi, \phi) \geq \beta^{(e)}(l_{c}^{(n)} \ldots l_{c}^{(n)}, \xi, \phi) \).

It is enough to prove for \( e = 2 \) since the same steps follow for the general case. For \( e = 2 \),
\[
\beta^{(2)}(P_{L_{1}^{(1)}}, P_{L_{2}^{(2)}}, \xi, \phi) = \int_{S_{e}} \beta^{(2)}(P_{L_{1}^{(1)}}, l_{c}^{(2)}, \xi, \phi) dP_{L_{2}^{(2)}}(l_{c}^{(2)})
\]
For any \( l_{c}^{(1)} \), let \( \xi_{c}^{(1)}(l_{c}^{(1)} - l_{c}^{(1)}) = \xi(l_{c}^{(1)} - l_{c}^{(1)}) + \xi(l_{c}^{(1)} - l_{c}^{(1)}) \) and
\[
\beta^{(2)}(P_{L_{1}^{(1)}}, l_{c}^{(2)}, \xi, \phi) = \beta^{(1)}(P_{L_{1}^{(1)}}, l_{c}^{(2)}, \xi, \phi) \geq \beta^{(1)}(l_{c}^{(1)} - l_{c}^{(1)}, \xi, \phi),
\]
which follows because \( l_{c}^{(1)} \) satisfies (4) for any decay function. Using the inequality obtained,
\[
\beta^{(2)}(P_{L_{1}^{(1)}}, P_{L_{2}^{(2)}}, \xi, \phi) \geq \int_{S_{e}} \beta^{(1)}(l_{c}^{(1)}, l_{c}^{(2)}, \xi, \phi) dP_{L_{2}^{(2)}}(l_{c}^{(2)})
\]
\[
= \beta^{(2)}(l_{c}^{(1)} - l_{c}^{(1)}, \xi, \phi) = \beta^{(1)}(P_{L_{2}^{(2)}}, l_{c}^{(2)}, \xi, \phi)
\]

where (29) and the inequality of (30) follow using the same steps used to obtain the first inequality, but now conditioning on \( \{L_{1}^{(1)} = l_{c}^{(1)}\} \) and the first conclusion is reached.

For the second conclusion, when \( E \) emitters are at \( l_{c}^{(1)} \), the conditional distribution of \( Z_{i,j} \) becomes \( F(z_{i,j}|E(\xi(l_{c}^{(n)} - l_{c}^{(n)}))) \), which is equivalent as having a single signal with amplitude multiplied by \( E \). Since \( F(z|x) \) is nonincreasing with \( x \), \( P_{1}[E = 1] = 1 \) provides the lowest probability of detection.

**References**


